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On the collapse analysis of single span masonry/stone arch bridges with fill interaction

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ABSTRACT: In this paper a comparison between two non – linear finite element models which have been developed for the analysis of masonry arches is presented. According to the first model, the geometry of the arch is divided into a number of unilateral contact interfaces which simulate potential cracks. Opening or sliding for some of the interfaces indicate crack initiation. The second model initially uses two-dimensional finite elements for the simulation of the arch. When tensile stresses appear, the corresponding elements are replaced by non linear gap elements which represent cracks. In both models the fill over the arch is taken into account. Moreover, the ultimate load and the collapse mechanism have been calculated by using a path-following (load incrementation) technique. Both models are developed and applied on a real scale masonry arch, so that the results can be compared.

1 INTRODUCTION

A masonry arch consists of masonry blocks and mortar joints. Blocks have high strength in compression and low strength in tension while mortar has generally low strength. The mentioned variation in the mechanical properties of the bridge's materials makes the study of them quite demanding and leads to the development of a number of theories in order to represent as accurately as possible, the real mechanical behaviour of the stone bridge.

One classical method developed for the study of stone arches was established by Heyman (1982). It is based on the assumption that an arch fails by the development of a collapse mechanism with four hinges. Several methods have then proposed for the assessment of the masonry arch. A part of them are related with the limit analysis of block structures with a frictional contact interface law. Melbourne and Gilbert (1995) confirmed that frictional assumptions are very important in multiring arches. Orduna and Lourenco (2005a, b) developed two- and three-dimensional models of discrete structures (like stone arches) and they took into account torsion failure mode. They also included in their study reinforcement elements. Other methods are based on the development of a finite element model in the framework of the incremental analysis. Crisfield (1984, 1985) proposed a model in which the arch is simulated with beam elements. He took into account the fill over the arch as well as the active and passive soil pressure induced by fill, by using non - linear, one dimensional elements. Lofti and Shing (1994) developed a discrete finite element model for the description of the mortar joints of masonry structures. They simulated mortar with interface elements with a non linear constitutive law. Molins and Roca (1998) used a three-dimensional finite element model for the investigation of the behaviour of stone arches. They applied the Mohr – Coulomb criterion for the shear failure of the masonry, and the perfect - plastic constitutive law for the simulation of the tensile failure mode. Cavicchi and Gambarotta (2005) simulated arches and piers with beam elements having zero tensile strength. For the fill they used two-dimensional plane strain finite elements with the Mohr - Coulomb failure criterion. For the arch - fill interaction they applied interface elements.

They presented applications in a single span as well as in a multi - span masonry bridge. From another point of view, Ng and Fairfield (2004) proposed a modification of the collapse mechanism method by considering the interaction between the arch deflections and the backfill pressures.

In this study a comparison between two models developed for the investigation of the mechanical behaviour of stone arches is presented. According to the first model, the ultimate failure load of a stone arch bridge is found by the usage of a discrete model formulation. In particular, the geometry of the structure is divided into a number of interfaces, perpendicular to the center line of the ring. Those interfaces are uniformly distributed along the arch. Unilateral contact law governs the behaviour in the normal direction of an interface, indicating that no tension forces can be transmitted in this direction. The behaviour in the tangential direction takes into account that sliding may or may not occur, by the usage of the Coulomb friction law. The backfill can be included in this model. It is simulated with two dimensional finite elements. The interaction between arch and fill is also simulated by a unilateral contact - friction interface. The either - or decisions incorporated in the unilateral contact and friction mechanisms make the whole mechanical model highly nonlinear. Due to the presence of non - differentiable functions within these models, they are characterized as nonsmooth mechanics models (Mistakidis and Stavroulakis 1998). For practical applications carefully tuned path - following iterative techniques are used for the numerical solution. Furthermore, the limit analysis problem is related to the solvability of the underlying mechanical problem using analogous theoretical results concerning the solvability of variational inequalities and complementarity problems (Stavroulakis *et al.* 1991). The main idea of the second model is to propose a simplified approximate procedure, easily understandable from the designers. In this framework, a non-linear incremental approach is adopted, which considers masonry as a no - tension material. The procedure is based on a simplified two-dimensional finite element discretization of the masonry bridge. The arch barrel is modelled with four nodes, plane elasticity finite elements and the fill with non-linear springs.

Results of the comparison between the two models will prove to be quite interesting, as they will demonstrate how a more complex finite element model interacts with a simplified model, both useful for the assessment of stone arches.

2 THE FEM MODEL WITH UNILATERAL CONTACT - FRICTION INTERFACES

2.1 Frictional - contact mechanics

The behaviour in the normal direction of an interface is described by the unilateral contact model. In particular, let us consider the boundary of an elastic body which comes in contact with a rigid wall. Let u be the single degree of freedom of the system, g be the initial opening and t^n be the corresponding contact pressure in case contact occurs. The basic unilateral contact law is described by the set of inequalities (1), (2) and by the complementarity relation (3), (Stavroulakis *et al.* 1991, Panagiotopoulos 1985)

$$h = u - g \leq 0 \Rightarrow h \leq 0 \quad (1)$$

$$-t^n \geq 0 \quad (2)$$

$$t^n(u - g) = 0 \quad (3)$$

Inequality (1) represents the non penetration relation, relation (2) implements the requirement that only compressive stresses (contact pressures) are allowed and equation (3) is the complementarity relation according to which either separation with zero contact stress occurs or contact is realized with possibly non - zero contact stress.

The behaviour in the tangential direction is defined by a static version of the Coulomb friction model. Two contacting surfaces start sliding when the shear stress at the interface reaches a maximum critical value equal to

$$t^t = \tau_{cr} = \pm \mu |t^n| \quad (4)$$

where t' , t'' are the shear stress and the contact pressure at a given point of the contacting surfaces respectively and μ is the friction coefficient. There are two possible directions of sliding along an interface, therefore t' can be positive or negative depending on that direction. Furthermore, there is no sliding if $|t'| < \mu |t''|$ (stick conditions). The stick - slip relations of the frictional mechanism can be mathematically described with two sets of inequalities and complementarity relations, similar to (1)-(3), by using appropriate slack variables (Mistakidis and Stavroulakis 1998).

2.2 Formulation and solution of the unilateral contact - friction problem

For the frictional - contact problem the Virtual Work equation is written in a general form

$$\int_V \mathbf{q} \cdot \delta \mathbf{e} dV = \int_S \delta \mathbf{u} \cdot \mathbf{t} dS + \int_V \delta \mathbf{u} \cdot \mathbf{f} dV + \int_{S'} \delta \mathbf{u} \cdot \mathbf{t}'' dS' + \int_{S'} \delta \mathbf{u} \cdot \mathbf{t}' dS' \quad (5)$$

where t'' and t' are the normal and tangential traction vectors on the actual contact boundary S' , \mathbf{q} is the stress tensor, $\delta \mathbf{e}$ is the virtual strain tensor, $\delta \mathbf{u}$ is the virtual displacement vector and \mathbf{t} , \mathbf{f} are the surface and body force vectors, respectively.

The nonlinearity in the unilateral contact problem is introduced by the variational inequality (Panagiotopoulos 1985)

$$\delta \mathbf{u} \cdot \mathbf{t}'' \leq 0 \quad (6)$$

and the nonlinearity in the frictional problem is introduced by the variational inequality

$$\delta \mathbf{u}_t \cdot \mathbf{t}' \leq \max (\delta \mathbf{u}_t \cdot \mathbf{t}'_{cr}, -\delta \mathbf{u}_t \cdot \mathbf{t}'_{cr}) \quad (7)$$

Here t'_{cr} is the vector of the critical shear stresses τ_{cr} , in the tangential direction of the interfaces. Relation (7) implies that no slip occurs when $|t'| < \tau_{cr} = \mu |t''|$ while slip starts when $t' = \tau_{cr}$. Lagrange multipliers are also used in the Principle of the Virtual Work to enforce sticking conditions. The arising set of the nonlinear equations is solved by the Newton - Raphson incremental iterative procedure, or by specialized algorithms (for linear or nonlinear complementarity or nonsmooth optimization problems).

For example, the frictionless unilateral contact problem takes the following matrix form

$$\mathbf{K}\mathbf{u} + \mathbf{N}^T \mathbf{r} = \mathbf{P}_o + \lambda \mathbf{P}, \mathbf{N}\mathbf{u} - \mathbf{g} \leq 0, \mathbf{r} \geq 0, (\mathbf{N}\mathbf{u} - \mathbf{g})^T \mathbf{r} = 0 \quad (8)$$

Equation (8.a) expresses the equilibrium equations of the unilateral contact problem, where for simplicity frictional terms are omitted. \mathbf{K} is the stiffness matrix and \mathbf{u} is the displacement vector. \mathbf{P}_o denotes the self - weight of the structure and \mathbf{P} represents the concentrated live load. \mathbf{N} is an appropriate geometric transformation matrix and vector \mathbf{g} contains the initial gaps for the description of the unilateral contact joints. Relations (8.b,c,d) represent the constraints of the unilateral contact problem for the whole discretized structure and are based on the local description given by relations (1), (2), (3). The enforcement of the constraints is achieved by using Lagrange multipliers. Thus, \mathbf{r} is the vector of Lagrange multipliers corresponding to the inequality constraints and is equal to the corresponding contact pressure ($-t''$). The problem described above is a nonsmooth parametric linear complementarity problem (LCP) parametrized by the one - dimensional load parameter λ . All required quantities can be calculated by using finite element techniques. Using path - following the solution of the problem can be calculated in the interval $0 \leq \lambda \leq \lambda_{failure}$, where $\lambda_{failure}$ is the value of the loading factor for which the unilateral contact problem does not have a solution. This is the limit analysis load. The analysis reported here has been completed within the ABAQUS code (see Drosopoulos *et al.* 2006 for further details).

3 A FEM MODEL WITH ADAPTIVE CRACK ELEMENTS

Various kinematic approaches have been developed in the literature (Faccio *et al.* 1995), where the load carrying capacity of the bridge is calculated from a limit condition in which an adequate number of hinges reduces the structure to a mechanism. It is implicitly assumed that hinges due to no-tension behaviour may be activated in the arch barrel, while the material may sustain unbounded, or very large, inelastic strains. In order to overcome these drawbacks, and

take into account the non-linear response of the masonry, complex load settings and the actual bridge geometry, more detailed models are needed such as the ones provided by non linear incremental two and three-dimensional FEM procedures (Crisfield 1984). A relatively recent development in the analysis of masonry arch bridges has been the use of finite element techniques. Towler *et al.* (1982) showed the potential of this general approach by computing load deflection curves and collapse loads for an arch modeled with one-dimensional beam elements to represent the arch barrel. The presence of fill was considered only as a dead load and no further soil-structure interaction effects were modeled. Crisfield (1985) later introduced non-linear spring elements in an attempt to model, in a simplified way, the lateral resistance from the soil. Some smeared, three-dimensional crack models have been used here in order to take into account the brittle behaviour of masonry, as they have been implemented in the computer code ANSYS. In such a model the cracking process is completely introduced via some constitutive (concrete-like) laws which affects the material stiffness at every integration point. These criteria requires the determination of some parameters that cannot be evaluated experimentally in a simple or reliable way.

A simplified numerical procedure for the limit analysis of masonry arch bridges is developed in order to take into account, and to evaluate, the arch-fill interaction. The procedure is based on a simplified two-dimensional finite element discretization of the masonry bridge. The arch is modeled by means of plane stress elastic elements and the fill is modeled by means of non-linear links. Interface (gap) elements are also applied on the connection between the fill area and the extrados of the arch. The main idea of the present approach is to propose a simplified approximate procedure, suitable for design engineers. The first researcher that introduced non-linear spring elements for the modelling of the fill material and for the modelling of the lateral resistance of the soil was Crisfield (1985). The spring stiffness were termed the sub-grade modulus, and were initially pre-compressed to the equivalent pressure at rest. The maximum horizontal pressure was limited by the active or passive pressures depending upon the type of movement of the arch. In this work gap elements are used in order to take into account the possible sliding between arch barrel and fill material. The non-linear behaviour of the arch barrel, modelled by two-dimensional four-nodes elastic elements, is reproduced by means of gap elements, based upon the original idea of Castigliano. At each load increment the stress behaviour of this plane element is checked at every joint, and if the stress is not admissible for the masonry material (that is, tensile stresses arise) the corresponding joints that connect the adjacent elements are substituted by contact elements. This is an iterative procedure, implemented inside ANSYS.

4 A THEORETICAL COMPARISON BETWEEN THE TWO APPROACHES

A significant difference between the two models is introduced by the fact that in the model with the gap elements the development of the cracked area is not imposed "a priori" but follows the actual behaviour of the arch when loading is increasing from zero to the final load. Of course the position of the gap elements is determined by the mesh size of the model. In contrast, the contact model uses fixed positions for the interfaces of the arch indicating that the positions of the potential cracks are imposed "a priori". The two models converge for fine discretizations.

Furthermore, the fill is simulated in two different ways. In the model with the gap elements, non linear springs are used. This simplified procedure is particular useful for a first quick assessment of the behaviour of the stone arch. This is attributed to the fact that springs contribute to a relatively simple and fast numerical solution. On the other hand, the contact model uses two dimensional finite elements with the Mohr - Coulomb failure criterion. This is a more complex way of simulating the fill over the arch. It gives information about the failure of the fill and a more realistic representation of it, but it demands more computational time and effort.

5 APPLICATION ON A REAL ARCH

The case study analyzed corresponds to the Prestwood bridge (see Fig. 1), a single-span bridge tested to collapse within the experimental research on masonry bridges supported by the Transport Research Laboratory (Page 1983).

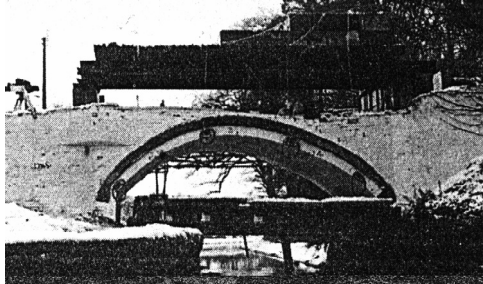


Figure 1 : Prestwood bridge before destructive test (DT)

This bridge has a span length equal to 6.55 m, with upper rise 1.42 m. The vault thickness is 0.22 m and comprises a single ring of bricks laid as headers; the fill depth at the crown is 0.165 m and the total bridge width is 3.800 m. The load is applied on a strip of the road surface along the full width of the bridge between the parapets at a point where the lower load value leading to failure was expected. The latter has been evaluated by assuming that the arch would fail as a four hinged mechanism and was, then, applied at the quarter-span of the bridge. The strip was 0.30 m wide in order to distribute the load and to avoid a premature failure of the fill. The load has been applied by means of hydraulic jacks, while the required reaction for the load was provided by the weight of concrete blocks on a steel frame above the bridge (see Fig. 2).

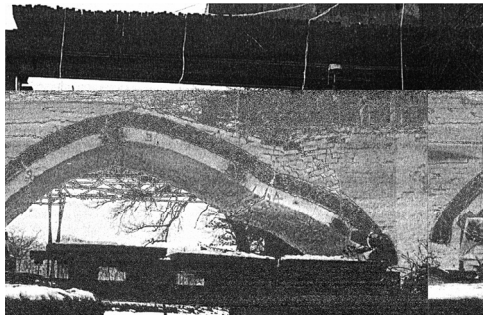


Figure 2 : Collapse mechanism

The fill density is 20 kN/m^3 and the masonry density is 20 kN/m^3 . The experimental collapse mechanism of the bridge is shown in Fig. 2 (Page 1983). Tab. 1 reports the parameters used in the analysis models. The vault collapse mechanism exhibits four hinges that are clearly visible in the picture; the mechanism developed with negligible material crushing. The arch mechanism constrains the fill region under the applied load to move downward and the fill at the other side of the bridge to move upward. The first visible sign of damage occurred at $P_f = 173 \text{ kN}$ and the experimental collapse load was $P_u = 228 \text{ kN}$.

5.1 Results from the unilateral contact - friction model

In the framework of the contact model, the fill is simulated with two-dimensional plane strain finite elements. The arch - fill interaction is taken into account, as well. In particular, the model of the arch with the interfaces developed in previous sections is used. The interaction between the arch and the fill is also simulated by a unilateral contact - friction interface. Failure of the fill material is representing by the Mohr – Coulomb failure criterion. Concerning the boundary

conditions, both the arch and the fill are initially considered to be fixed to the ground. In particular, the horizontal as well as the vertical boundaries of the fill are fixed.

For the mechanical properties mentioned here the collapse load which is obtained by the contact model is $P_u = 225.4\text{KN}$ which is comparable with the experimental value (228KN). A four hinges mechanism arises in the arch, which is the same with the experimental one. In Fig. 3 the collapse mechanism obtained by the contact model is shown, suitably scaled to make deformation visible. The brown color in the fill indicates the region where yield occurs. Both the arch and the fill move downward in the left - hand - side of the structure (e.g. the loaded side). As a consequence, the side of the bridge opposite to the loading moves upward (the four hinges collapse mechanism of the arch pushes the fill upwards in this side). The hinge number (4) of Fig. 3 does not appear very close to the right springing but is moved toward the left side. The presence of the fill in Fig. 3 is responsible for the offset of the hinge's position.

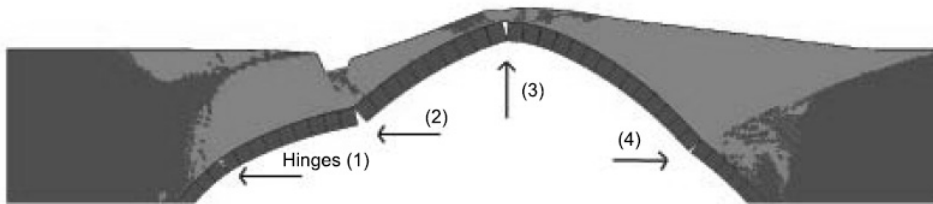


Figure 3 : Four hinges collapse mechanism from the unilateral contact-friction m

5.2 Results from the adaptive crack element model

The arch barrel structural elements are modeled with plane42 elements. Gap elements are added at the area where tensile stresses arise, due to the applied increasing load. Interaction between fill material and arch barrel have been taken into account by means of non linear springs.

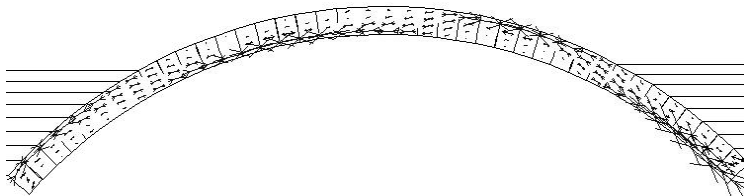


Figure 4 : Thrust line before collapse

Some parametric analyses have been performed in order to evaluate the influence of the elastic material properties of the arch barrel elements (i.e. Young modulus). The analyses have shown that, by neglecting second order effects connected with geometric nonlinearities, the elastic material properties are not important for the evaluation of the ultimate load (Loo 1995).

Table 1 : Material properties.

Fill density	γ_{fill}	20 kN/m ³
Fill cohesion	c_{fill}	10 kPa
Fill angle of internal friction	ϕ	37°
Masonry ring density	γ_m	20 kN/m ³
Masonry compressive strength	σ_m	7.7 N/mm ²

The loaded side of the arch moves downward, and the arch mechanism pushes the left side; as a consequence, the fill over the left side is moved upward. The first hinge on the left side of the arch does not develop at the springing, as it happens in the arch where the fill is heavy but not resistant; the presence of the fill constrains the hinge to move upward. This behaviour

agrees with the experimental result shown in Fig. 2. The numerical critical load is 225.7 kN, a value which is very close to the experimental collapse load $P_u = 228$ kN.

The strengthening effects of the fill correspond to a higher exploitation of the strength resources of the arch; in particular, the development of the collapse mechanism predicted by the limit analysis requires the development of plastic compressive strains higher than those predicted by simplified models without fill resistance that could be no longer consistent with the masonry behaviour. Fig. 5 shows principal stresses; close to the area where hinge are arising there is a local crushing of masonry material but, again, the value reached by the tensional state on the arch at the collapse denote that collapse is due to a development of a mechanism rather than a global crushing. Fig. 4 shows the thrust line, which is adjacent to the ring of the arch on four section denoting that an hinge is opened in that point. The section of the arch that first reaches the compressive elastic limit is located under the area where the load is applied. This result agrees with the experimental behaviour of the bridge and shows that this analysis can be applied with a good approximation.

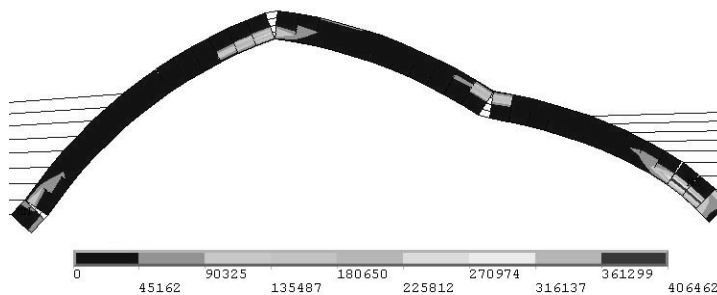


Figure 5 : Principal Stress σ_{11} (kg/m^2) before collapse

5.3 Comparison between the two methods

An interesting comment can be made by observing the collapse mechanism obtained by the two models. In particular, it seems that the position of the hinge number (4) in the right side of the contact model, does not coincide with the corresponding hinge in the model with the gap elements (first hinge in the left side of the model). The latter is developed very near to the springing while the hinge of the contact model arises at some distance from it. This is attributed to the fact that in the contact model both the vertical and the horizontal boundaries of the fill are fixed. In contrast, in the model with the gap elements the fill is simulated with horizontal springs which permit a movement in the horizontal direction. This corresponds to free vertical boundaries, for the contact model. Indeed, if vertical boundaries of the contact model become free, the collapse mechanism will be slightly modified and the particular hinge will be in the same position for both models, e.g. very close to the springing. This is shown in Fig. 3. The limit load in this case is equal to 223.4 kN. As the boundary conditions have been changed, fill parameters have to be changed too, in order to obtain the above mentioned limit load. Consequently, angle of internal friction has been considered to be equal to 37° , cohesion equal to 10 kPa and dilation angle equal to 33° .

6 CONCLUSIONS

A large number of arch bridges built in Europe during the 19th century are still in service. They were built according to codes of practice and design criteria developed for the 19th century loads. Most of these bridges that are still in their original configuration are nowadays subjected to heavier loads and sometimes show signs of deterioration. Due to their importance for the transportation systems, especially for the European railway network, a simplified tool for a reliable estimation of their actual load carrying capacity is needed. The structural analysis of masonry structures has always been a very demanding task. The numerical results shown in this paper demonstrate that nonlinear, and in particular unilateral models, can be used for the reliable prediction of the ultimate (limit) load and the collapse of masonry bridges. Comparison

with experimental results shows that both methods are able to access with good confidence the ultimate load for masonry arch structures. With the contact model, where the fill is modelled with plane element, is possible also to obtain information about the arising stress and strain state of the fill material. The unilateral model cannot supply detailed information about the fill because of the simplified assumption. Anyway the method is proved to be user-friendly since it can be developed by making use of the programming facilities of commercial FE codes and, in spite of some approximations on model parameters, it is proved to give good precision and it could be an interesting alternative to the standard nonlinear facilities of commercial FEM codes, which are generally related to concrete-like materials.

One should mention that unilateral phenomena arise in a larger number of heritage and monumental structures, like domes and vaults (Leftheris *et al.* 2006). The here proposed techniques can, in principle, be extended to cover more general structures as well. Detailed analysis of three-dimensional structures and the effect of dynamical loads will be reported in the future elsewhere.

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