A GLOSA SYSTEM WITH STOCHASTIC SIGNAL SWITCHING TIMES

Diploma Thesis

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Abstract

The main goal of this work is to generate optimal trajectories for vehicles crossing a signalized junction, with traffic signals function in real-time (adaptive signals). Adaptive signals decide their next switching time according to the prevailing traffic conditions, and that is why the information of the switching is unknown beforehand (a-priori). Green Light Optimal Speed Advisory (GLOSA) systems use information of the current state and timing of a traffic signal in order to guide the driver (or an automated vehicle) all the way to the traffic light by giving appropriate speed advise, which ensures that the vehicle will cross the traffic signal at green and with minimum fuel consumption and emissions.

In previous works the problem of producing fuel-optimal vehicle trajectories for vehicles approaching an intersection, with traffic lights operating in both cases of fixed and stochastic switching times was considered. However, there was an assumption that the traffic signal is initially red and turn to green, which means that only half traffic light cycle was considered.

In this work, the aforementioned problem is extended considering a full traffic light cycle, meaning that in the case of fixed switching times the traffic light switches from red to green and vice versa, while in case of unknown switching times, the traffic signal's cycle consists of four phases: a certain green phase, in which the vehicle can freely pass; an uncertain green phase, in which there is a probability that the traffic light will extend its duration or it will turn to red; a certain red phase that the vehicle cannot pass and; an uncertain red phase, in which there is a probability that the red will be extended or it will turn to green. For the first case, the problem is formulated as an optimal control problem and is solved analytically via PMP (Pontryagin's Maximum Principle). In the second case, the traffic light switching times is not known and depends for example on the prevailing traffic conditions. In such cases, there are typically time-windows of admissible switching times for each stochastic phase; hence probability distributions of the switching times can be derived within these time-windows, e.g., based on statistics from past signal switching activity. Thus, the problem was cast in the format of a stochastic optimal control problem, which was solved numerically using Stochastic Dynamic Programming (SDP) techniques.

Περίληψη

Ο κύριος στόχος αυτής της διπλωματικής εργασίας είναι να δημιουργήσει βέλτιστες τροχιές για οχήματα που διασχίζουν μια σηματοδοτημένη διασταύρωση, με λειτουργία προσαρμοσμένων σημάτων κυκλοφορίας σε πραγματικό χρόνο. Τα προσαρμοστικά σήματα αποφασίζουν τη μεταγωγή τους σύμφωνα με τις επικρατούσες κυκλοφοριακές συνθήκες και γι' αυτό οι πληροφορίες της αλλαγής είναι άγνωστες εκ των προτέρων. Τα συστήματα Green Light Optimal Speed Advisory (GLOSA) χρησιμοποιούν πληροφορίες για την τρέχουσα κατάσταση και χρονική στιγμή ενός σήματος κυκλοφορίας για να καθοδηγήσουν τον οδηγό (ή ένα αυτόνομο όχημα) μέχρι το φανάρι, υπολογίζοντας τη βέλτιστη τροχιά και προφίλ ταχύτητας για μια αρχική κατάσταση (θέση και ταχύτητα) και μια τελική κατάσταση, η οποία διασφαλίζει ότι το όχημα θα περάσει το φωτεινό σηματοδότη κυκλοφορίας στο πράσινο με ελάχιστη κατανάλωση καυσίμου και εκπομπών ρύπων.

Σε προηγούμενες εργασίες εξετάστηκε το πρόβλημα της παραγωγής βέλτιστων τροχιών οχημάτων με βάση την κατανάλωση καυσίμου, για οχήματα που πλησιάζουν σε διασταύρωση, με φανάρια που λειτουργούν και στις δύο περιπτώσεις των σταθερών και στοχαστικών χρόνων εναλλαγής από πράσινο σε κόκκινο. Ωστόσο, υπήρχε η υπόθεση ότι το σήμα κυκλοφορίας είναι αρχικά κόκκινο και γίνεται πράσινο, πράγμα που σημαίνει ότι λήφθηκε υπόψη μόνο ο μισός κύκλος του φαναριού.

Στην εργασία αυτή, το προαναφερθέν πρόβλημα επεκτείνεται λαμβάνοντας υπόψη έναν πλήρη κύκλο φωτεινού σηματοδότη, δηλαδή στην περίπτωση σταθερών χρόνων εναλλαγής το φανάρι αλλάζει από κόκκινο σε πράσινο και αντίστροφα, ενώ σε περίπτωση στοχαστικών χρόνων εναλλαγής, ο κύκλος του σήματος αποτελείται από τέσσερεις φάσεις: μία βέβαιη πράσινη φάση, στην οποία το όχημα μπορεί να περάσει ελεύθερα από την διασταύρωση. Μια αβέβαιη πράσινη φάση, στην οποία υπάρχει πιθανότητα το φανάρι να παρατείνει τη διάρκειά του ή να γίνει κόκκινο. Μια βέβαιη κόκκινη φάση που το όχημα δεν μπορεί να περάσει και τέλος μια αβέβαιη κόκκινη φάση, στην οποία υπάρχει πιθανότητα το κόκκινο να παραταθεί ή να γίνει πράσινο. Για την πρώτη περίπτωση, το πρόβλημα διατυπώνεται ως πρόβλημα βέλτιστου ελέγχου και επιλύεται αναλυτικά μέσω PMP (Pontryagin's Maximum Principle). Στη δεύτερη περίπτωση, οι χρόνοι εναλλαγής των φωτεινών δεν είναι γνωστοί και εξαρτώνται για παράδειγμα από τις επικρατούσες συνθήκες κυκλοφορίας. Σε τέτοιες περιπτώσεις, υπάρχουν συνήθως χρονικά παράθυρα αποδεκτών χρόνων εναλλαγής για κάθε στοχαστική φάση. Ως εκ τούτου, οι κατανομές πιθανοτήτων των χρόνων εναλλαγής μπορούν να προκύψουν μέσα σε αυτά τα χρονικά παράθυρα, π.χ., με βάση στατιστικά στοιχεία από προηγούμενη δραστηριότητα εναλλαγής του σήματος. Έτσι, το πρόβλημα διατυπώθηκε με τη μορφή ενός προβλήματος στοχαστικού βέλτιστου ελέγχου, το οποίο λύθηκε αριθμητικά χρησιμοποιώντας τεχνικές Στοχαστικού Δυναμικού Προγραμματισμού (SDP).

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Chapter 1: Introduction

1.1 Prolegomena and Related Work

Nowadays the energy resources are dying off and the environment's protection is vital. Therefore, it is essential for transportation systems to operate with increased fuel efficiency, to offset the increasing price of energy. Road vehicles are widely used and are an integral part of many people's lives. Fuel efficiency will help the fuel economy, which leads to less fixed expenses for the driver and environmental protection in times that the earth and atmosphere cannot withstand any additional burden. The automotive industry has developed new engines, designs, lighter chassis and control systems that reduce fuel consumption. In addition, new intelligent transportation systems (e.g., adaptive traffic signals) have contributed to decreased congestion and fuel economy.

The circulation on the roads is being controlled by traffic signals in order to keep safe the crossing of vehicles at junctions. In advance to what is said before, safety can be applied by traffic lights that force vehicles to decelerate when there is a red light, and after switching to green to accelerate so the vehicle can continue to its final destination. This kind of movement affects the fuel consumption of the vehicles involved. Aiming in the reduction of the stops and delays, the optimization of the traffic lights operation is key to not have the speed of the vehicle fluctuating. Consequently, a wide range of algorithms have been presented and utilized to set up the traffic lights in a manner that serves the above purpose. In one hand with fixed-time traffic signals, the plans are derived offline for various hours of the day depending on past data and implemented by appropriate optimization codes (thus the traffic light switching time is known beforehand). In the other hand real-time (or traffic responsive or adaptive) traffic signal control algorithms take into account the current state in the road and junction to evaluate the best possible switching during their cycle. In this case the next switching is not known before the switching decision is actually made and based on the followed strategy, the control update period can vary from one second to one signal cycle. In addition, it is significant to note that the real-time techniques and approaches are considered to be more advanced and effective than the fixed time signal implementations. In recent years, fuel consumption is a very important aspect and an optimization criterion for a signal control system is being developed.

Regard a vehicle that proceeds toward a green traffic light at a predefined and known speed. If the vehicle keeps approaching the traffic light with a constant speed and it switches to red before the vehicle has passed it, then the vehicle will be obligated to stop at the traffic signal and accelerate again after the switch from red to green. On the other hand, if the vehicle accelerates enough during the green phase, it can pass the traffic light without immobilizing and also if the vehicle makes the necessary adjustments in his speed, it can cross the traffic light when the traffic light will be again green without any stop. Those decisions regarding the speed that have to be made by the driver or the automated vehicle while reaching the traffic signal may be addressed by appropriately designed systems. In the early stages, systems were displaying on road-side dynamic advisory speed signs that would give the necessary information to the driver to cross the junction while the traffic light is green (Leersum, 1995). However, advances and innovations in the transportation industry have made it possible to transmit the current state and timing of the traffic signal to the vehicle itself or in its installed apps. Taking into account the previous, it is feasible to guide the driver or the automated vehicle from the moment the vehicle is in a predefined distance from the traffic light, so it will follow the optimal trajectory (fuel and emission wise) to the traffic signal. The above-mentioned systems are often referred to as Green Light Optimal Speed Advisory (GLOSA) systems (Stahlmann et al., 2016).

Looking into the fixed time case that the switching time is known a-priori, e.g., the information can be transmitted through messages by a signal controller. This has been addressed by various studies. In one of them, speed profiles have been compared with their energy demand (Richter, 2005). Nevertheless, in this restricted set of profiles there is a deficiency of mathematical justification that the speed profile is for sure optimal. To decrease fuel consumption and emissions, ruled-based algorithms have been utilized in a number of works (Katsaros et al., 2011.; Ma et al., 2018; S'anchez et al., 2006) to derive advisory speed profiles. Rule-based algorithms are capable of delivering sub-optimal results, especially when the kinematics of the vehicle are not taken into consideration (no acceleration just speed is included). Methods that have considered the vehicle kinematics are more suitable to derive fuel-optimal speed profiles. The aforementioned approach has been studied(Lawitzky et al., 2013), solving analytically through Pontryagin's minimum principle (PMP) the optimal control problem.

A more realistic and thus more complicated situation is when real-time traffic signals with short control update are considered. In this case prior knowledge of the following switching time is not accessible (not even with the signal controller). An estimate or a probabilistic distribution are two ways to get the information about the next signal switching. In a study (Mobisys 11 Conference Committee, 2011), a system named SignalGuru depends only on mobile phone data collection to predict the switching of the signal, and it does not require communication with the infrastructure. The mobile phones capture the present traffic signals through their cameras, predicting their future alternations, since they have communicated among themselves first and have created switching patterns. Regarding SignalGuru, for fixed traffic signals and real-time traffic signals, the average time for the estimates turn out to be within $0.66s$ and $2.45s$, respectively. GLOSA systems that require a known switching time can utilize such estimates and with suitable extensions that will minimize the estimation inaccuracy, a proxy will be achieved.

A different approach to obtain the prediction for the next signal switching time can be also generated in form of a probability distribution within a short-term future time-window, using prior signal operations that were taken over an adequate long rolling horizon (Mahler & Vahidi, 2012). In the previous approach the probability distribution is applied heuristically and not optimally, to time-weigh the objective function in the deterministic optimal problem that is being solved through a dynamic programming algorithm. Looking further, the use of the probabilistic distribution of the next signal switching in the case of a stochastic optimal control problem has been introduced (Lawitzky et al., 2013). The solution of the problem is made through a discrete stochastic dynamic programming (SDP) algorithm. Nevertheless, the beforementioned approach can lead to non-commensurable vehicle trajectories and kink solutions because the formulated optimal control problem expands only up to the point the switching time end up known (the cost till the final state of the vehicle has not been considered).

This thesis studies the problem of extracting fuel-optimal vehicle trajectories (for both known and unknown switching times) for a vehicle advancing a traffic signal (for a full traffic light circle). For the case of known switching, the problem is solved analytically through PMP. Afterwards for the case of the stochastic switching time with a known probability distribution, the problem is treated as a stochastic optimal control problem, which is solved numerically applying SDP techniques. On the contrary to earlier works, the whole traffic cycle is considered (green, uncertain green extension, red, uncertain red extension) and the problem extends until a pre-selected final state after the traffic signal. The problem after the switching from red to green occurs or the vehicle crosses the traffic light in the green phase, returns to the analytical solution via PMP of the fixed time switching time case. The resulting solution is consistent and non-dependent to the actual switching time.

1.2 GLOSA system

The fluctuation of speed plays a major role in fuel consumption. In urban areas where the roads are smaller, the volume of the vehicle is higher and there are many intersections, it is frequent to deal with congestion. As a result, the vehicle has higher speed fluctuations and immobilizations. To face this problem, traffic lights and signals have been adapted. To maximize the functionality of the traffic signals new methods of optimization have been proposed, as well as advisory methods for drivers or autonomous vehicles to adjust their behavior. Connected vehicles technology gives a two-way wireless communication that allows vehicle-to-infrastructure and vehicle-to-vehicle connections(Stahlmann et al., 2016).

The ultimate aim of Green Light Optimal Speed Advisory (GLOSA) systems is to lower CO_2 emissions and to avoid unnecessary stopping in intersection approach scenarios by giving speed advices to drivers based on current and future traffic light signal phase timings. Consequently, the high fuel consumption that derives from stop-and-go driving will be minimized (Stahlmann et al., 2016).

GLOSA can be separated into two different approaches, where is the single and multi-segment GLOSA. In a single-segment GLOSA traffic signals are considered independently, i.e., the system provides vehicles with the optimal speed for the segment ahead of the nearest traffic signals. In a multi-segment GLOSA several signals in a sequence on a vehicle's route are taken into account, that is, vehicles receive speed advices for a set of segments ahead of the vehicle and then it calculates an optimal set of speed before entering the first segment (Seredynski et al., 2013).

Systems like GLOSA will perform an important role in the future of international transportation and it is crucial to be compatible for every type of adaptive traffic lights. Adaptive traffic signals adjust the timing of their green light cycles to match current traffic conditions on the ground. They are constantly collecting data about approaching vehicles and creating new timing sequences to match them. First of all, Video cameras and sensors collect information about the vehicles approaching an intersection. Then, software analyzes this information and creates a customized timing sequence in real time. The software communicates this sequence to coordinated signals up and down the corridor, so that they all function in sync with each other. As a result, they move traffic along faster and with fewer stops. Signals are constantly being reprogrammed to maximize the green light length and allow the most cars through. Multiple intersections are coordinated, so that traffic can move freely throughout the corridor, rather than encountering frequent starts and stops.

In general, GLOSA functionality is based on two message types: SPAT and MAP. A Signal Phase and Timing Message (SPAT) informs about current state, current phase and next phase for each lane of an intersection, Map Data Messages (MAP) provide information about the topology of an intersection such as number of lanes and turning restrictions. SPAT and MAP messages are transmitted by single-hop broadcast. The procedure for a driver to receive speed advice is as follows. To begin with, the vehicle receives at least one message of every type and links them using the intersection's singular identification included in the messages that have been sent. Then, the GLOSA application generates geometry from the MAP message to match the vehicle's position and determine the corresponding lane number. When the lane of the vehicle is revealed, signal phases and timing data related to the specific lane are matched.

The vehicles that can communicate with the infrastructure transmit information to the application. For the approaching vehicle to the traffic signal, the system proposes a range of feasible speeds, so the vehicle will pass the traffic light while it is in its green phase. That way the vehicle will not immobilize. The GLOSA will send advice speed only if it is necessary and the driver or the autonomous vehicle do not already have a speed that reaches the requirements. If the previous is true, then the system examines the following incoming vehicle (Stevanovic et al., 2013).

1.3 Goals of this work

This thesis main goal is to generate optimal trajectories for vehicles crossing a signalized junction, with traffic signals operating in real-time (adaptive signals) and considering a full traffic light cycle. Adaptive signals decide their switching according to the prevailing traffic conditions, and that is why the information of the switch is unknown beforehand (a-priori). Green Light Optimal Speed Advisory (GLOSA) systems use information of the current state and timing of a traffic signal in order to give an appropriate speed advice to the driver (or an automated vehicle) all the way to the traffic light by giving appropriate speed advise, which ensures that the vehicle will cross the traffic signal at green, which leads to a more efficient driving fuel wise and thus environmentally friendly.

Chapter 2: GLOSA Problem Formulation

OPTIMAL CONTROL PROBLEMS AND SOLUTIONS FOR GLOSA WITH KNOWN OR UNKNOWN SIGNAL SWITCHING TIME

In this segment the GLOSA approach is going to be described in detail for the cases of known or unknown signal switching time.

2.1 Optimal Control Problem with Known Signal Switching Time

This section analyzes in detail the proposed GLOSA approach when the traffic signal switching time is known a priori. A formulation for a vehicle's trajectory having a kickoff from an initial state to arrive at a final state during a free time horizon, taking into consideration a traffic light at a pre-specified position.

2.1.1 Problem Formulation

Assume a vehicle travelling from an initial state $x_0 = [x_0, v_0]^T$, where x_0 stands for a given initial position and v_0 for a given initial speed. The goal is to lead the vehicle to reach at a fixed final state $x_e = [x_e, v_e]^T$ during a free (but weighted) time horizon t_e, with x_e and v_e being the final position and speed respectively, but also taking into account the position x_1 of the traffic light and the known time t_1, which is the time that the signal switches from red to green. The assumption here is that the vehicle at time 0 finds the traffic light in red (certain red phase). The objective of the vehicle is to make the appropriate modifications in acceleration (control variable), so as a result to achieve the minimum fuel consumption, while satisfying the initial and final conditions x_0 and x_e , in addition to the intermediate constraint of the traffic signal.

It is important to mention that the fixed final position x_e is a pre-specified position after the traffic light and the final speed v_e is the velocity of the vehicle in this position. These assumptions are made to ensure that the problem will not be myopic, and the best trajectory will be selected. If the final position was selected at the traffic signal ($x_e = x_1$) as in some previous works, then (Lawitzky et al., 2013):

- If also the final speed is fixed, that would possibly rule out fuel-optimal trajectories and the speed in the position of the traffic light will be different.
- Otherwise, if the final speed is not fixed, there would be various trajectory candidates. Some will be with low speed at the x_1 position (that is more costly (fuel wise) because of the future need of acceleration) and others with higher speed (lower need of acceleration).

The minimization problem that is described above is formulated as an optimal control problem. The vehicle's kinematics are described as follows:

$$\dot{x} = v \tag{1}$$

$$\dot{v} = a \tag{2}$$

where a is the vehicle's acceleration which is the control variable. The objective is to control the acceleration in a way that will guide the system from the given initial condition $x_0 = [x_0, v_0]^T$ to the fixed final condition $x_e = [x_e, v_e]^T$ within time t_e, while taking into consideration the minimization of the criterion:

$$J = w \cdot t_e + \frac{1}{2}\int_0^{t_e} a^2 dt \tag{3}$$

Furthermore, the traffic light constraint $t_s \geq t_1$ must be satisfied, where t_s is the time the vehicle crosses from the traffic light x_1 ($x(t_s) = x_1$). The cost term a^2 in the minimization cost criterion was proved to be an excellent proxy for fuel-minimizing vehicle trajectories (Typaldos et al., 2020). In (Typaldos et al., 2020) was shown that the square of acceleration derives results that are a very good approximate of a complex fuel consumption model. Note also that the final time t_e is free (flexible problem formulation), that is, it can apply for different initial positions (far or close to the traffic light), initial speeds, signal switching times, and takes a penalty based on the parameter w. This penalty ensures that the final time t_e will not take excessive values. The larger the penalty w is, the lower the value of the final time t_e will be. The higher the value of w is, the vehicle will tend to accelerate so it reaches sooner the final position (the acceleration cost will increase accordingly). The next figure shows how w affects the final time t_e and the acceleration cost.

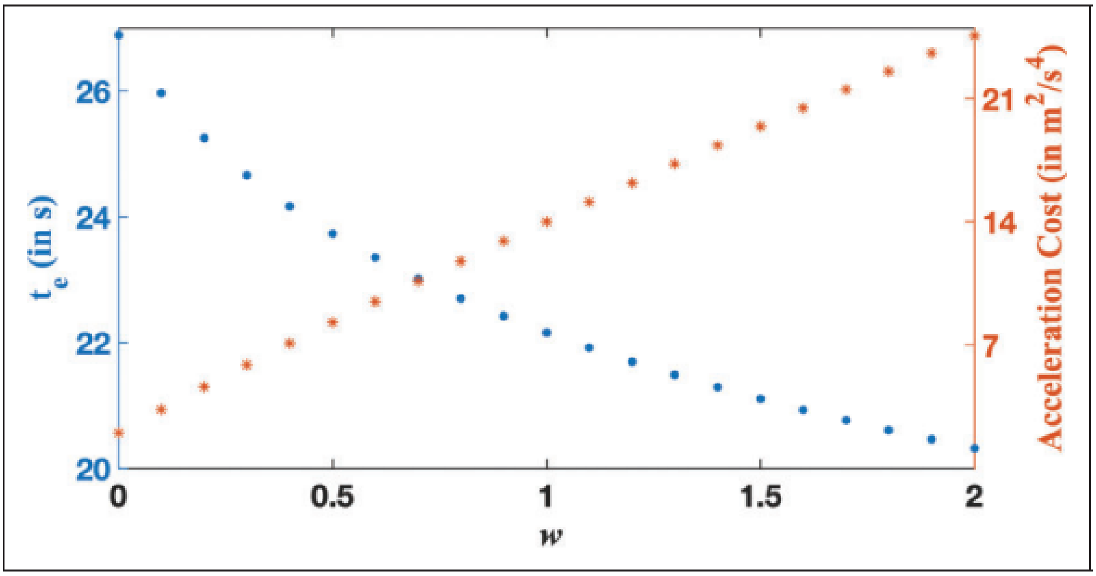

Figure 1. Optimal final time t_e (blue dotted marks) and acceleration cost (red star marks) versus the weight w.

Furthermore, bounds are applied to the speed v and acceleration a of the vehicle.

2.1.2 Analytical Solution

To advance at the presentation of the analytical solution, it is essential to introduce the following Hamiltonian function (Pontryagin,2018)(Typaldos et al., 2020):

$$H(x, \lambda) = \varphi(x, a) + \lambda^T f(x, u) \tag{4}$$

where λ is the co-state vector for the corresponding state equations and $\varphi(x, a) = \frac{1}{2}a^2$ is the cost term we saw before in the objective function (Equation 3). For this problem, the Hamiltonian function reads as follows:

$$H(v, a, \lambda_1, \lambda_2) = \frac{1}{2}\alpha^2 + \lambda_1 v + \lambda_2 \alpha \tag{5}$$

In addition, the necessary conditions of optimality have to be met:

$$\dot{x} = \frac{\partial H}{\partial \lambda_1} = v \tag{6}$$

$$\dot{v} = \frac{\partial H}{\partial \lambda_2} = \alpha \tag{7}$$

$$\dot{\lambda_1} = -\frac{\partial H}{\partial x} = 0 \tag{8}$$

$$\dot{\lambda_2} = -\frac{\partial H}{\partial v} = -\lambda_1 \tag{9}$$

$$\frac{\partial H}{\partial a} = 0 \tag{10}$$

The boundary conditions also have to be satisfied. Furthermore, a new boundary condition for optimality is introduced (Pontryagin, 2018)(Typaldos et al., 2020):

$$H(t_e) + w = 0 \tag{11}$$

There are two problems to be considered, the unconstrained problem (UP) and the constrained problem (CP). The UP does not include the green light constraint ($t_s \geq t_1$). In this case, the Equations 8-10 are drowned out to produce the linear-in-time optimal acceleration solution for the UP.

$$a(t) = c_1 t + c_2 \tag{12}$$

By integrating Equation 12 one time and two times we get the speed and position solutions respectively.

$$v(t) = \frac{1}{2}c_1t^2 + c_2t + c_3 \tag{13}$$

$$x(t) = \frac{1}{6}c_1t^3 + \frac{1}{2}c_2t^2 + c_3t + c_4 \tag{14}$$

where c_1, c_2, c_3, c_4 are integration constants and with the optimal final time t_e can be specified through the solution of a system of five algebraic equations, while taking into account the initial and final state and the final time condition (Equation 11).

Proceeding with the "constrained" problem (CP). In this case, the vehicle has to be at the traffic light at some given time t_c ($x(t_c) = x_1$). The conditions Equations 6-11 stay unaltered, but with the addition of the condition that the co-state $\lambda_1(t)$ may be discontinuous in-between the time t_c (Typaldos et al., 2020). The last-mentioned condition and the Equations 8-10 generate a continuous two branch piece wise linear optimal acceleration solution for this specific CP.

$$a(t) \begin{cases} c_1t + c_2 & 0 \leq t \leq t_c^- \\ c_5t' + c_6 & t_c^+ \leq t \leq t_e \end{cases} \tag{15}$$

By integrating Equation 15 we get the corresponding speed and position solutions.

$$v(t) = \begin{cases} \frac{1}{2}c_1t^2 + c_2 + c_3 & 0 \leq t \leq t_c^- \\ \frac{1}{2}c_5{t'}^2 + c_6t' + c_7 & t_c^+ \leq t \leq t_e \end{cases} \tag{16}$$

$$x(t) = \begin{cases} \frac{1}{2}c_1t^3 + \frac{1}{2}c_2t^2 + c_3t + c_4 & 0 \leq t \leq t_c^- \\ \frac{1}{6}c_5t'^3 + \frac{1}{2}c_6{t'}^2 + c_7t' + c_8 & t_c^+ \leq t \leq t_e \end{cases} \tag{17}$$

where $t' = t - t_c$ and $c_1, c_2, c_3, \ldots., c_8$ are integration constants, which along with the final time t_e can be described as a system of nine algebraic equations. This system includes the initial and final states, the final time condition Equation 11, the continuity conditions for control and states $a(t_c^-) = a(t_c^+), v(t_c^-) = v(t_c^+)$, $x(t_c^-) = x(t_c^+)$ and the intermediate condition $x(t_c) = x_{max}$.

J_{UP} and $J_{CP}(t_c)$ (a function of t_c) are the optimal objective values of UP and CP problem respectively. $J_{CP}(t_c)$ is equal or greater to the $J_{UP} \forall t_c$ because CP is UP problem with additional constraints. The minimum value of $J_{CP}(t_c)$ is obtained for $t_c = t_{S,UP}$, where $t_{S,UP}$ is the time when the vehicle in the UP reaches the traffic light x_1. The $J_{CP}(t_c)$ increases monotonically for $t_c \geq t_{S,UP}$. Considering the above, for traffic signal position x_1 and switching time t_1 the GLOSA problem may be solved as:

1. Solve UP: if $t_{S,UP} \geq t_1$, the GLOSA problem is solved, as the UP solution does not violate the green-light constraint; else
2. Solve CP with $t_c = t_1$ to obtain the GLOSA solution.

To obtain the solution of the UP and CP of the five and the nine algebraic equations respectively, symbolic differentiation tools can be used to retrieve the solutions. The solution of UP and CP can be obtained in fragments of seconds of computation time once the switching time becomes known or there can be a memory on the vehicle system that will have stored the ideal control based on the pre-execution of all the possible combinations of position, speed and control (acceleration-deceleration). The vehicle's trajectory can also continuously update (in a model predictive control [MPC] loop) to include various obstacles and further driving limitations.

The optimal value (Equation 3) of the deterministic GLOSA problem is denoted as $J_{DG}^*(\boldsymbol{x_0}, t_1)$ (the problem is depended on $\boldsymbol{x_0}, t_1$).

In this formulation, it is assumed that the traffic light at time 0 is red. All the same can be applied in the other case that the traffic signal is green at time 0 when the vehicle enters the studied distance from the traffic light. The switching time from green to red is known (T_r). The $J_{CP}(t_c)$ increases monotonically for $t_c \leq t_{S,UP}$. Consequently, this case of initial green is described below:

1. Solve UP: if $t_{S,UP} \geq t_1$ or $t_{S,UP} \leq T_r$, the GLOSA problem is solved, as the UP solution does not violate the green-light constraint; Else
2. Solve CP twice, first with $t_c = t_1$ or $t_{S,UP} \leq T_r$ and second with $t_c = T_r$, yielding the respective optimal values $J_{CP}(t_1)$ and $J_{CP}(T_r)$. The solution with the smaller objective value is the GLOSA solution of the generalized problem.

2.2 Stochastic Optimal Control Problem with Uncertain Signal Switching Time

In this case the traffic light switching time is unknown or the switching is depended on short-term decisions because of the prevailing traffic conditions (real time signals). Thus, the problem is formulated as a stochastic optimal problem, which is solved numerically using SDP (Stochastic Dynamic Programming) variants. Note that, the stochastic approach takes into account the analytical solution of the deterministic GLOSA optimal control problem.

2.2.1 Problem Variables and State Equations

The kinematics of the vehicle that describe the discrete time SDP algorithms with time step T are:

$$x(k+1) = x(k) + v(k)T + \frac{1}{2}a(k)T^2 \tag{18}$$

$$v(k+1) = v(k) + a(k)T \tag{19}$$

where $x(k)$ is the vehicle position and $v(k)$ is his speed (both at discrete times $k = 0,1, \dots.$). It is also known that $t = kT$. In addition, the $a(k)$ which is the control variable is constant over each time period k. Both the control variable and the state variable are bounded between values as shown below:

$$\boldsymbol{x(k)} \in \boldsymbol{X} = [\boldsymbol{x_{min}}, \boldsymbol{x_{max}}] \tag{20}$$

$$a(k) \in U = [a_{min}, a_{max}] \tag{21}$$

where x_{min}, x_{max} and a_{min}, a_{max} being the lower and upper bounds respectively of the variables mentioned above. As for the unknown switching times k_1, k_2 of the traffic light, it is assumed that there are known ranges for each phase, for green $k_{min}^G < k_1 < k_{max}^G$ and for red $k_{min}^R < k_2 < k_{max}^R$ of possible switching times, where k_{min}^R, k_{min}^G and k_{max}^R, k_{max}^G are the minimums and the maximums possible switching times respectively for each phase. These bounds are applied because most real time traffic lights work with pre-fixed minimum and maximum for each of the phases (green and red). As a result, the switch from green to red and the opposite are happening between a default time window. The total time horizon $[0, K]$ is subdivided in 4 parts as seen below:

- Part 1: $[0, k^G_{min} - 1]$ (certain green)
- Part 2: $[k^G_{min}, k^G_{max} - 1]$ (uncertain green extension)
- Part 3: $[k^G_{max}, k^R_{min} - 1]$ (certain red)
- Part 4: $[k^R_{min}, k^R_{max} = K]$ (uncertain red extension)

2.2.1.1 Red Phase

The problem formulation for the red phase (Typaldos et al., 2020) needs a virtual state $x^R(k)$ that helps to reflect properly the stochasticity of the problem.

$$x^R(k+1) = \begin{cases} 1 - z^G(k) & \text{if } x^R(k) = 0 \\ x^R(k) z^R(k) & \text{else} \end{cases} \tag{22}$$

where $z_r(k)$ is a discrete stochastic variable defined as:

$$z^R(k) = \begin{cases} 0 \; if \; switch \; occurs \; at \; time \; k+1 \\ 1 \; else \end{cases} \tag{23}$$

Considering the previous two equations, the virtual state $x^R(k)$ is going to be equal to 0, if the green light has not yet switched until time $k-1$, or if the red-green switching took place at time not later than k, or equal to 1, if the green light has switched, but the red light has not yet switched until $k-1$. The system knows each time kT if switching has occurred or not in the last time period $[(k-1)T, kT]$, as $x^R(k)$ is measurable.

The stochastic variable $z^R(k)$ receives values based on a time dependent probability distribution $p_r(z|k)$ and is independent from its previous values ($z^R(k-1), z^R(k-2), \ldots$). The $p_r(z|k)$ probability is being calculated based on data and statistics of previous signal switching activity. In addition, for signal switching within a time window, an a-priori discrete probability distribution $P^R(k)$ ($k^R_{min} < k_1 < k^R_{max}$) is taken into account, where $\sum_{k^R_{min}}^{k^R_{max}} P(k) = 1$.

For the case $k \leq k^R_{min} - 1$ that there is no switching:

$$p_r(0|k) = 0 \; for \; k < k^R_{min} - 1 \tag{24}$$

For $k = k^R_{min}$, a signal switching might happen with an a-priori probability $\mathrm{P}(k^R_{min})$:

$$p_r(0\big|k^R_{min} - 1) = \mathrm{P}(k^R_{min}) \tag{25}$$

As shown in (Lawitzky et al., 2013), if the traffic signal does not switch at time $k = k^R_{min}$, then the probabilities of switching are building up compared with the respective a-priori distribution, as long as the switching in the time window does not happen. These probabilities can be measured via "crop-and-scale", that is, the a-priori probability $\mathrm{P}(k^R_{min})$ follows a uniform distribution and increases the probabilities of the remaining discrete times withing the time window. The switching probabilities update every timestep until the switching. Below is the crop-and-scale formula that is described above for $k^R_{min} < k_2 < k^R_{max}$ and for any a priori distribution $P(k)$:

$$p_r(0|k) = P(k+1)\left[1 + \frac{\sum_{\kappa=k^R_{min}}^{k} P(\kappa)}{\sum_{\kappa=k+1}^{k^R_{max}} P(\kappa)}\right] \tag{26}$$

2.2.1.2 Green Phase

The problem formulation for the green phase (GLOSA with Uncertain Green and Red Phases, 2022) also needs a virtual state $x^G(k)$ that helps to reflect properly the stochasticity of the problem.

$$x^G(k+1) = x^G(k) \cdot z^G(k) \tag{27}$$

$$x^G(0) = 1 \tag{28}$$

where $z^G(k)$ is a binary stochastic variable

$$z^G(k) = \begin{cases} 0\ if\ switch\ occurs\ at\ time\ k+1 \\ 1\ else \end{cases} \tag{29}$$

The virtual state $x^G(k)$ is either equal to 0, if the traffic light has switched at time k or earlier, or equal to 1 if the switching has not yet happened until time $k-1$. The $x^G(k)$ is assumed measurable, which leads at the conclusion that at each time kT it is known if the switching occurred or not within the previous time period $[(k-1)T, kT]$. The vehicle in the green phase can overcome the bound x_1 (the traffic light position) during the time period $[(k-1)T, kT]$, at which the switching happened. This can occur because the vehicle applied the last change in the acceleration at time $(k-1)T$, earlier than the switching.

As the corresponding in red phase, the stochastic variable $z^G(k)$ is independent from its values in the past ($z^R(k-1), z^R(k-2), \ldots$). It takes values according to a time dependent probability distribution $p_g(z|k)$. The required probabilities $p_g(z|k)$ for the stochastic variable $z^G(k)$ are being calculated using the same method as in red phase, that is, using crop-and-scale with an a-priori discrete probability distribution $P^G(k)$ ($k^G_{min} \le k \le k^G_{max}$) with the sum of the probabilities equal to 1.

2.2.2 Objective Criterion

- Red Phase

The stochastic problem has similar cost criterion as in the deterministic case (Equation 3), with the difference that the exact value of the criterion depends on the stochastic variable's realization, and therefore its expected value is minimized.

$$J = E\left\{w\ \cdot t_e\ + \frac{1}{2}\int_0^{t_e} a^2 dt\right\} \tag{30}$$

where the expectation refers to the stochastic variable $z^R(k)$, $k = k^R_{min}, \dots, k^R_{max} - 1$. Note that, the problem becomes a deterministic GLOSA problem when the traffic light switches and becomes known at the time k_2. The vehicle is at state $x(k_2)$ at the switching and optimal cost-to-go is $J^*_{DG}[x(k_2), k_2]$ (escape cost).

The Equation 30 transforms in the Equation 31 after discrete time notation and by the principle of optimality.

$$J = E\left\{\frac{1}{2}\sum_{k=k^R_{min}}^{k_2-1} a(k)^2 + J^*_{DG}[x(k_2), k_1]\right\} \tag{31}$$

To formulate properly the cost criterion, the virtual state variable $x^R(k)$ and the stochastic variable $z^R(k)$ are assimilated. In addition, the state $x(k_2)$ from the state Equations 18,19 is substituted as a function of the state and control of the previous time period (Bertsekas, 2005). This leads to a reformed objective function, which is the following

$$J = E\left\{x^R(k)\sum_{k=k^R_{min}}^{k^R_{max}-1}\left[\frac{1}{2}\, a(k)^2 + [1 - z^R(k)]\, J^*_{DG}[x(k_2), k_2]\right]\right\} \tag{32}$$

The ordinary stochastic optimal control problem consists by Equations 18-26 and 32. The recursive Bellman equation (Bertsekas, 2005) for $k^R_{min} < k < k^R_{max}$ is reformed as shown in the Equation 33 below, by denoting the optimal cost-to-go function by $V[x(k), x^R(k), k]$.

$$\begin{aligned} V[x(k), x^R(k), k] &= \min_{a(k)\in U}\left\{E\left\{\frac{1}{2}\, a(k)^2 + [1 - z(k)]\cdot J^*_{DG}[x(k_2), k_1] + V[x(k+1), x^R(k)z(k), k+1]\right\}\right\} \\ &= \min_{a(k)\in U}\left\{\frac{1}{2}\, a(k)^2 + p_r(0|k)\cdot J^*_{DG}[x(k_2), k_2] + [1 - p_r(0|k)] \right. \\ &\qquad \left. \cdot V[x(k+1), 1, k+1]\right\} \end{aligned} \tag{33}$$

with $V[x(k_{max}^R),1,k_{max}^R] = 0$ as boundary condition. Notice that the minimum is wanted only with respect to the control $a(k) \in U$.

- Green Phase

For the formulation of the objective function in the green phase, it is best to subdivide the time horizon of Parts 1 and 2. In this way, it will be easier to extract the required objective functions.

At time $k = k_{max}^G - 1$, the traffic light has either already switched to red, thus $x^G(k_{max}^G - 1) = 0$ and the RBE in this case is the same as for Part 3, or the switch will definitely happen at k_{max}^G, as it is the last possible green time. In this particular case, we have $x^G(k_{max}^G - 1) = 1$, but again $x^G(k_{max}^G) = 0$ since we must have $z^G(k_{max}^G - 1) = 0$. In some scenarios it may be favorable for the vehicle to have the traffic lights (x_1) crossed, with speed and acceleration between the predefined bounds of them, sooner than the switching at k_{max}^G takes place, that is within the times $[(k_{max}^G - 1)T, k_{max}^G T]$. In this case, the vehicle will have a position $x(k_{max}^G)$ greater than x_1. The cost from the moment the traffic lights switch till the final state x_e is calculated as the escape cost $J_{DG}^*[x(k_{max}^G), k_{max}^G]$. To get such trajectories, the term $J_{DG}^*[x(k_{max}^G), k_{max}^G]$ is included in the objective function of the problem for all positions $x(k_{max}^G) > x_1$ that are that are accessible with respect to the defined bounds in speed and acceleration. More specifically, we introduce a function $\sigma(x(k+1))$, which is equal to 1 if the next conditions are met $x(k+1) > 0 \ and \ v(k+1) < v_{max}$ or else equal to 0. Considering the previous function, we modify the escape cost function

$$J_{DG}^{\sigma}{}^*(x(k_{max}^G), k_{max}^G) = \sigma\big(x(k_{max}^G)\big) \cdot J_{DG}^*(x(k_{max}^G), k_{max}^G) \tag{34}$$

that will be used in the objective function. All the above functions can be expressed with the arguments $x(k), a(k)$ with the use of the state equations.

On the other hand, it may not be beneficial even with the escape cost to cross the traffic light during the times $[(k_{max}^G - 1)T, k_{max}^G T]$ for some states $x(k_{max}^G - 1)$. In this case, the vehicle will have optimal cost $V[x(k_{max}^G), 0, k_{max}^G]$ at time k_{max}^G.

The RBE for $k = k_{max}^G - 1$ is as follows:

$$
\begin{aligned}
&V[x(k_{max}^G - 1), x^G(k_{max}^G - 1), k_{max}^G - 1] \\
&\quad = \min_{u(k_{max}^G - 1) \in U} \left\{ E\left\{ \frac{1}{2}\, a(k_{max}^G - 1)^2 + J_{DG}^{\sigma}{}^{*}(x(k_{max}^G), k_{max}^G) \right.\right. \\
&\quad \left.\left. + V[x(k_{max}^G), x^G(k_{max}^G), k_{max}^G] \right\}\right\} \\
&\quad = \min_{u(k_{max}^G - 1) \in U} \left\{ \frac{1}{2}\, a(k_{max}^G - 1)^2 \right. \\
&\quad + J_{DG}^{\sigma}{}^{*}(x(k_{max}^G - 1), a(k_{max}^G - 1), k_{max}^G) \\
&\quad \left. + V[x(k_{max}^G - 1), a(k_{max}^G - 1), 0, k_{max}^G] \right\} \qquad (35)
\end{aligned}
$$

The function V is set 0 at non-admissible states where the function $J_{DG}^{\sigma}{}^{*} \neq 0$.

Within Part 2 for previous time steps $k = k_{max}^G - 2, k_{max}^G - 3, k_{max}^G - 4, \ldots, k_{min}^G$, exist again two cases. In the first case, the traffic light has already switched to red, hence we have $x^G(k) = 0$ and the cost function $V[x(k + 1), 0, k + 1]$ for time $k + 1$. In the second case, the traffic light has not switched yet to red, hence we have $x^G(k) = 1$ and the cost function for time $k + 1$ is subdivided into two cases:

- with probability $p_g(0|k)$, the traffic light will switch at $k + 1$, hence $x^G(k + 1) = 0$ and the cost function V for $k + 1$ is the same as in case 1.
- with the complementary probability $(1 - p_g(0|k))$, the traffic light will not switch at $k + 1$, hence $x^G(k + 1) = 1$ and the cost function V will be $V[x(k + 1), 1, k + 1]$.

The modified escape cost is taken into account in both sub-cases, because the vehicle may benefit by crossing the traffic light.

The RBE in regard the above is:

$$V[x(k), x^G(k), k] = \min_{u(k)\in U} \left\{ E\left\{ \frac{1}{2}\ a(k)^2 + x^G(k)\ J_{DG}^{\sigma\ *}(x(k+1), k+1) + V[x(k+1), x^G(k+1), k+1] \right\}\right\}$$

$$= \min_{u(k)\in U} \left\{ E\left\{ \frac{1}{2}\ a(k)^2 + x^G(k)\ J_{DG}^{\sigma\ *}(x(k), a(k), k+1) + V[x(k)a(k), x^G(k)z^G(k), k+1] \right\}\right\}$$

$$= \min_{u(k)\in U} \left\{ \frac{1}{2}\ a(k)^2 + x^G(k)\ J_{DG}^{\sigma\ *}(x(k), a(k), k+1) + \left(1 - x^G(k) + p_g(0|k)\ x^G(k)\right) V[x(k), a(k), 0, k+1] + \left(1 - p_g(0|k)\right) x^G(k)\ V[x(k), a(k), 1, k+1]) \right\} \quad (36)$$

The function V is set 0 at non-admissible states where the function $J_{DG}^{\sigma\ *} \neq 0$.

At Part 1 for time $k_{min}^G - 1$, we are at the certain green phase, hence the traffic light cannot be red, and we have $x^G(k_{min}^G - 1) = 1$. In the next time steps, we stand in uncertain green phase, so it is possible for the traffic light to switch to red in the next time step k_{min}^G or later. Based on the mentioned above, the RBE is as follows:

$$V\left[x\left(k_{min}^G - 1\right), 1, k_{min}^G - 1\right] = \min_{u(k_{min}^G - 1)\in U} \left\{ \frac{1}{2}\ a\left(k_{min}^G - 1\right)^2 + J_{DG}^{\sigma\ *}\left(x\left(k_{min}^G - 1\right), a\left(k_{min}^G - 1\right), k_{min}^G\right) + p_g\left(0\middle|k_{min}^G - 1\right) V\left[x\left(k_{min}^G - 1\right), a\left(k_{min}^G - 1\right), 0, k_{min}^G\right] + \left(1 - p_g\left(0\middle|k_{min}^G - 1\right)\right)\ V\left[x\left(k_{min}^G - 1\right), a\left(k_{min}^G - 1\right), 1, k_{min}^G\right]) \right\} \quad (37)$$

Eventually, for the remaining time of the Part 1, that is for $k = k_{min}^G - 2, k_{min}^G - 3, \ldots, 0$, we have $x^G(k_{min}^G) = 1$ and it is sure that in later time steps no switching will occur. The corresponding RBE for this case is:

$$V[x(k),1,k] = \min_{u(k)\in U}\left\{\frac{1}{2}\ a(k)^2 + {J_{DG}^{\sigma}}^*(x(k),a(k),k+1) + V[x(k),1,k+1]\right\} \quad (38)$$

- United objective function and recursive Bellman Equation

To obtain a formally proper cost criterion, the stochastic variables $z^G(k), z^R(k)$ and virtual variables $x^G(k), x^R(k)$ are used, and, similarly to (Typaldos, et al., 2020), this yields the objective function as follows

$$J = E\left\{\sum_{k=0}^{k_{max}-1}\left[\frac{1}{2}a(k)^2 + \left(x^G(k) + \left(1 - z^R(k)\right)x^R(k)\right)\cdot {J_{DG}^{\sigma}}^*[\boldsymbol{x}(k),a(k),k+1]\right]\right\} \quad (39)$$

The recursive Stochastic Bellman Equation (SBE) has four corresponding parts. Starting from k_{max}^R, we need to move backwards, calculating the function V (optimal cost-to-go) step-by-step. The SBE for the generalized problem reads as follows

$$\begin{aligned}
&V[\boldsymbol{x}(k),x^G(k),x^R(k),k] = \\
&= \min_{u(k)\in U}\left\{E\left\{\frac{1}{2}\ a(k)^2 + \left(x^G(k) + (1 - z^R(k))x^R(k)\right)\cdot {J_{DG}^{\sigma}}^*(\boldsymbol{x}(k+1),k+1)\right.\right. \\
&\qquad\left.\left. + V[\boldsymbol{x}(k+1),x^G(k+1),x^R(k+1),k+1]\right\}\right\} = \\
&= \min_{u(k)\in U}\left\{E\left\{\frac{1}{2}\ a(k)^2 + \left(x^G(k) + (1 - z^R(k))x^R(k)\right)\cdot {J_{DG}^{\sigma}}^*(\boldsymbol{x}(k),k+1)\right.\right. \\
&\qquad\left.\left. + V[\boldsymbol{x}(k),a(k),x^G(k)z^G(k),x^R(k)z^R(k),k+1]\right\}\right\} = \\
&= \min_{u(k)\in U}\left\{\frac{1}{2}a(k)^2 + \left(x^G(k) + p_r(k)x^R(k)\right)\cdot {J_{DG}^{\sigma}}^*(\boldsymbol{x}(k),k+1)\right. \\
&\qquad + \left(p_g(k)x^G(k) + (1 - p_r(k))x^R(k)\right)\cdot V[\boldsymbol{x}(k),a(k),0,1,k+1] \\
&\qquad\left. + \left(\left(1 - p_g(k)\right)x^G(k)\right)\cdot V[\boldsymbol{x}(k),a(k),1,0,k+1]\right\}
\end{aligned} \quad (40)$$

where

$${J_{DG}^{\sigma}}^*(\boldsymbol{x}(k+1),k+1) = \sigma\left(\boldsymbol{x}(k+1)\right)\cdot J_{DG}^*(\boldsymbol{x}(k+1),k+1) \quad (41)$$

$$\sigma\big(\boldsymbol{x}(k+1)\big) = \begin{cases} 1 & \text{if } x(k+1) > \ x_1 \text{ and } v(k+1) \leq v_{max} \text{ and } k < k_{max}^G \\ \infty & \text{if } x(k+1) > \ x_1 \text{ and } v(k+1) > v_{max} \text{ and } k < k_{max}^G \\ 1 & \text{if } \boldsymbol{x}(k+1) \in \boldsymbol{X} \text{ and } k \geq k_{max}^G \\ \infty & \text{if } \boldsymbol{x}(k+1) \notin \boldsymbol{X} \text{ and } k \geq k_{max}^G \\ 0 & \text{else} \end{cases} \tag{42}$$

2.2.3 Numerical Solution Algorithm

To make possible the application of the Discrete Stochastic Dynamic Programming for numerical solution of the above problem, it is necessary for the state and control variables to be discretized. Computational time and memory requirements are highly affected by the level of the discretization. In addition, the accuracy of the solution is strongly impacted by the level of the discretization. Taking into account the previous criteria, the ideal trade-off is made to have reasonable computation system while the solution quality is acceptable.

Specifically for the problem's discretization the discrete time interval T is equal to 1s. Afterwards, assuming Δ to be a general discretization interval for the problem's variables. Setting the discretization interval of the acceleration equal to Δ ($\Delta\alpha = \Delta$), then the discretization interval of the speed can be assumed to be $\Delta v = \Delta\alpha$ because speed and acceleration intervals are equivalent (Equation 19). Observing the Equation 18, the discretization interval for the position is

$$\Delta x = \frac{1}{2}\Delta \cdot T^2 \tag{43}$$

Taking into consideration the above, if $x(k), v(k), a(k)$ are discrete points, then from Equation 18 and 19 it is concluded that $x(k+1), v(k+1)$ are discrete points as well. Based on that, assume

$$x(k) = n\Delta x \tag{44}$$

$$v(k) = m\Delta v = m\Delta \tag{45}$$

$$a(k) = l\Delta a = l\Delta \tag{46}$$

where $n,\ m,\ l$ are positive integers. Replacing Equations 43-46 to 18

$$x(k+1) = n\Delta x + m\Delta \cdot T + \frac{1}{2} l\Delta \cdot T^2 = \frac{1}{2} n\Delta \cdot T^2 + m\Delta \cdot T + \frac{1}{2} l\Delta \cdot T^2$$
$$= \frac{1}{2}\Delta \cdot T^2 \left(n + \frac{2}{T} m + l\right)$$
$$= \Delta(n + 2m + l) \tag{47}$$

The above proves that $x(k+1)$ is a discrete point. For $v(k+1)$ the same is concluded through the substitution of Equation 43-46 to Equation 19.

At this point, it is implemented the discrete SDP algorithm to obtain an optimal closed loop control law $a(k)^* = R[x(k), k]$, which will return the optimal acceleration $a(k)$ for any possible given state $\boldsymbol{x(k)}$ and time k. Starting from an initial state and time and having as a goal a final state, a full trajectory of the vehicle can be obtained by applying the optimal acceleration (control) values that were draw out from the discrete SDP algorithm. The discrete SDP algorithm can be visualized as a 3D grid, as displayed in the following Figure 2. The grid extends in time over the horizon $[0, k_{max}]$ and the two-dimensional state space with position x in $[x_{min}, x_{max}]$ and speed v in $[v_{min}, v_{max}]$. Each node matches to a discrete state point of $x(k)$, and each red connecting line corresponds to a discrete control $a(k)$ that shows the transition from $x(k)$ to $x(k+1)$ according to the state equations.

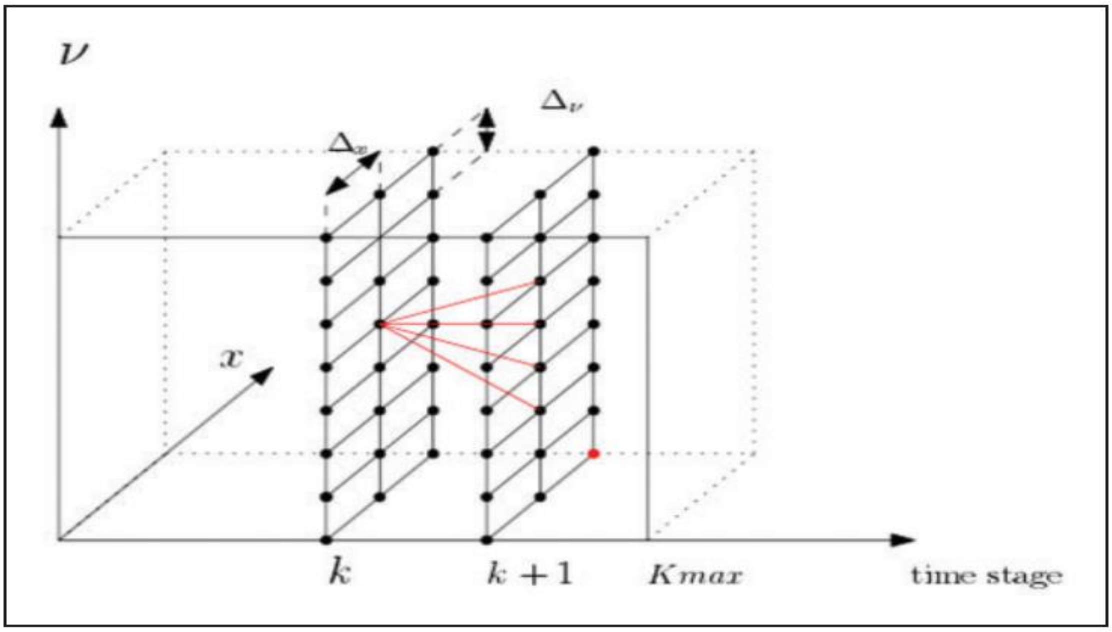

__Figure 2__. Illustration of the 3-D grid of time and state space.

Chapter 3: Results and Conclusions

3.1 Results

In this chapter are being demonstrated the results/findings of the investigated scenarios, which came through the implementation of the DDP algorithm for a whole traffic light cycle.

First of all, the key variables of the GLOSA problem are being explained briefly below:

- x_0: initial position of the vehicle in meters
- x_1: position of the traffic light in meters
- x_e: final position of vehicle in meters
- v_0: initial speed of the vehicle in m/s
- v_e: target final speed of the vehicle m/s
- $[0, k_{min}^G - 1]$: certain green time period in seconds
- $[k_{min}^G, k_{max}^G - 1]$: uncertain green extension time period in seconds
- $[k_{max}^G, k_{min}^R - 1]$: certain red time period in seconds
- $[k_{min}^R, k_{max}^R = K]$: uncertain red extension time period in seconds
- $[x_{min}, x_{max}]$: position bounds in meters
- $[v_{min}, v_{max}]$: speed bounds in m/s
- $[u_{min}, u_{max}]$: control bounds in m/s^2
- T_{min}^r: The time beginning of the uncertain red phase in seconds
- T_{max}^r: The end of the uncertain red phase in seconds
- T_{min}^g: The time beginning of the uncertain green phase in seconds
- T_{max}^g: The end of the uncertain green phase in seconds
- Tg_s: The actual switching time from green to red in the uncertain green phase
- Tr_s: The actual switching time from red to green in the uncertain red phase

From the above it is concluded that we have $T_{min}^r = k_{min}^R$, $T_{max}^r = k_{max}^R$, $T_{min}^g = k_{min}^G$, $T_{max}^g = k_{max}^G - 1$.

The scenarios of the SDP approach that were examined to study the behavior of the vehicle are presented in the table below.

Scenario	v_0	x_1	x_e	Tg_s	Tr_s	T^g_{min}	T^g_{max}	T^r_{min}	T^r_{max}
1	3,8,12	150,300,350	220,370, 420	10,15,18, 30	60	10	30	40	60
2	12	300	370	10-30	60	10	30	40	60

Table 1. *The full traffic light cycle examined scenarios.*

Stochastic GLOSA

In this section, the results of the discrete SDP approach are demonstrated. As it is shown in the above matrix there is a variety of scenarios as far as the initial speed (v_0), the position of the traffic light (x_1), the targeted final position (x_e) and the lengths of the phases (certain green, uncertain green, certain red, uncertain red). The states and control bounds are set $[x_{min}, x_{max}] = [0, 150]$m, $[0,300]$m, $[0,350]$m, $[v_{min}, v_{max}] = [0,16] m/s$, $[a_{min}, a_{max}] = [-2,1]\ m/s^2$. The discretization properties are as they were introduced before, because the time step T is set to 1 second. The switching time window from green to red and from red to green are as it follows:

Scenario	from Green to Red	from Red to Green
1	$[k^G_{min}, k^G_{max} - 1] = [10,30]$	$[k^R_{min}, k^R_{max} = K] = [40,60]$
2	$[k^G_{min}, k^G_{max} - 1] = [10,30]$	$[k^R_{min}, k^R_{max} = K] = [40,60]$

Table 2. *Time windows of possible traffic light switching.*

Furthermore, the discretization is $\Delta = \Delta\alpha = \Delta v = 0.125$, and therefore $\Delta x = 0.0625\ m$. The decision on the discretization is based on results of various discretization intervals (Figure 3). If we take a close look at the figure, we can detect that when the discretization interval is getting smaller, the cost is also reducing, and the CPU-time is heightening. The previous is justified since the smaller the discretization interval, the larger the number of grid points and as well as the grid point connections. Having the discretization interval (Δ) equal to 0.125, returns the best achievable result as far as the cost, but the CPU-time is large but not prohibitive.

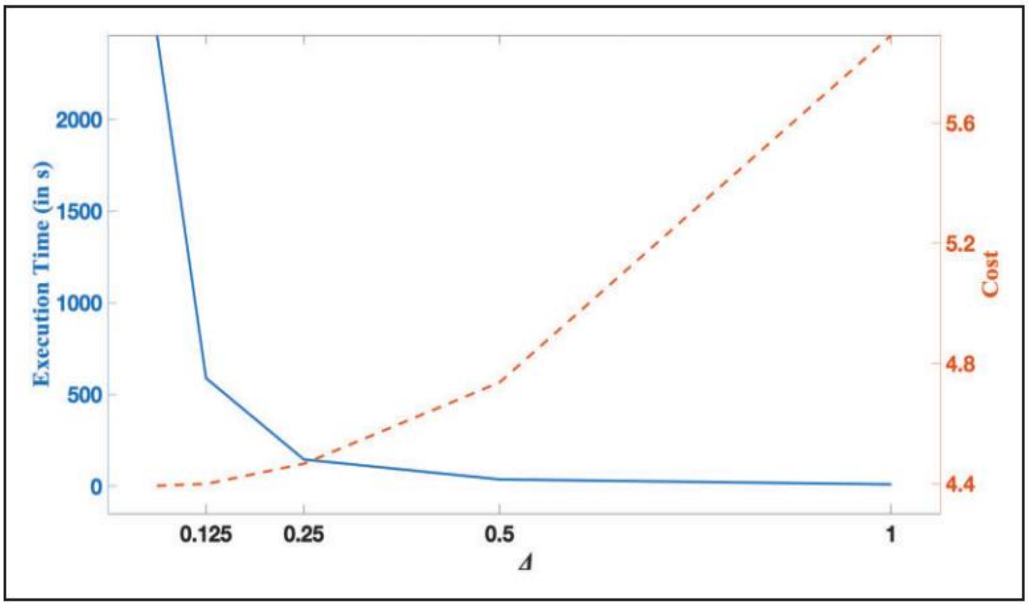

Figure 3. *Cost (red dotted line) and execution time (blue line) versus discretization interval Δ.*

3.1.1 Switching the traffic light distance x_1 and the actual switching time T_{g_s} with the same phase distribution (**Scenario 1**)

In the first scenario, the vehicle's trajectory, speed and acceleration is examined based on the initial speed, the position of the traffic light and the actual switching time Tg_s. The tested positions of the traffic light are those of 150, 300 and 350 meters from the initial position of the vehicle. In addition, the considered actual switching time Tg_s is set to 10s, 15s, 18s, 30s in the phase format that is seen in Table 1 above. The examined initial speeds v_0 for the above cases are $3.0m/s$, $8.0m/s$ and $12.0m/s$, so a range of speeds are covered. The final position x_e in all the cases is 70 meters further from the traffic light and the desired velocity is $v_e = 11m/s$ in the final state.

3.1.1.1 Traffic light at $\boldsymbol{x_1 = 300m}$

In the first sub-case, the trajectories with the traffic light at the 300 meters and with initial speed $3m/s$ is similar for the examined different actual switching times Tg_s at $10s, 15s, 18s$ as well as for $v_0 = 8m/s$ and $Tg_s = 10s$. In this case, the vehicle accelerates at first (in the certain green phase and as the actual switching time gets bigger the vehicle's acceleration extends into the uncertain green phase). This acceleration is due to the fact that the vehicle tends to pass the traffic light while it is green. This occurs because the algorithm does not know when the actual switching time will occur and follows an a-priori "crop-and-scale" probability $\mathrm{P}(k^R_{min})$ as shown in Equation 26. Afterwards, the vehicle decelerates after the switching to red, because it understands that it has to follow the optimal path to the traffic light until it switches again to green. In the last phase (uncertain red phase) the vehicle accelerates to pass the traffic light at the last possible time step. At that time, it is guaranteed that the traffic light will switch from red to green and as a result the vehicle will not have to immobilize and have to fluctuate its speed again.

For initial velocity $v_0 = 12.0m/s$ and actual switching time $10s$ and for $v_0 = 8.0m/s$-$Tg_s = 15s$ the vehicle increases speed until the switch from green to red and then it decelerates till the last time steps of the certain red phase where it immobilizes. This happens because the vehicle has covered most of the distance to the traffic light and it has not yet switched to green. It stays at that state in the start of the uncertain red phase. In this last phase of the full traffic light cycle, the vehicle then reaccelerates and reaches a speed that will enable it to reach at the traffic light at the last time step of the uncertain red phase, where it will be able to continue his course to the final state.

The pairs of values $v_0 = 12.0m/s\text{-}Tg_s = 15s$ and $v_0 = 8.0m/s\text{-}Tg_s = 18s$ have similar trajectories with the above-mentioned case with the difference that the vehicle immobilizes almost at the traffic light and so it only starts to accelerate again at the last moments of the uncertain red phase. This is a result of the initial speed combined with the actual switching time.

For $v_0 = 12.0m/s\text{-}Tg_s = 18s$ the trajectory is similar with the previous pairs of values, but it reaches the traffic light faster and stays immobilized for the whole certain and uncertain red phase. It reaccelerates again after the switching to green.

For actual switching time $Tg_s = 30s$ and initial velocities $v_0 = 3m/s, 8m/s, 12m/s$ the vehicle presents almost the same behavior. In all cases, the vehicle has the time to pass the traffic light in the uncertain green phase before the switch to red. The lower the initial speed, the bigger the acceleration. This way, the vehicle can reach a speed that will enable it to pass before the switch. In all cases the vehicle decelerates at the last time steps before the overtake of the traffic light, because it does not need further boost in its speed, considering the maximum time window in the current phase setup and the already covered distance.

The table below presents the fuel consumption based on the ARRB model and the final time t_e that the vehicle reaches the final position (370 meters in this case).

$v_0 - Tg_s$	Fuel (ml)	t_e (s)
3.0m/s-10s	65.2412	67.7667
8.0m/s-10s	66.9197	68.4627
12.0m/s-10s	65.2428	69.2255
3.0m/s-15s	69.8595	68.7593
8.0m/s-15s	71.4472	69.6474
12.0m/s-15s	73.6055	74.2366
3.0m/s-18s	73.8619	69.6305
8.0m/s-18s	77.246	74.2366
12.0m/s-18s	76.1527	75.5951
3.0m/s-30s	46.2444	35.6088
8.0m/s-30s	41.3671	31.9191
12.0m/s-30s	35.1281	28.8663

Table 3. *Fuel consumption (ARRB model) and final time for a combination of initial velocity and actual switching time*

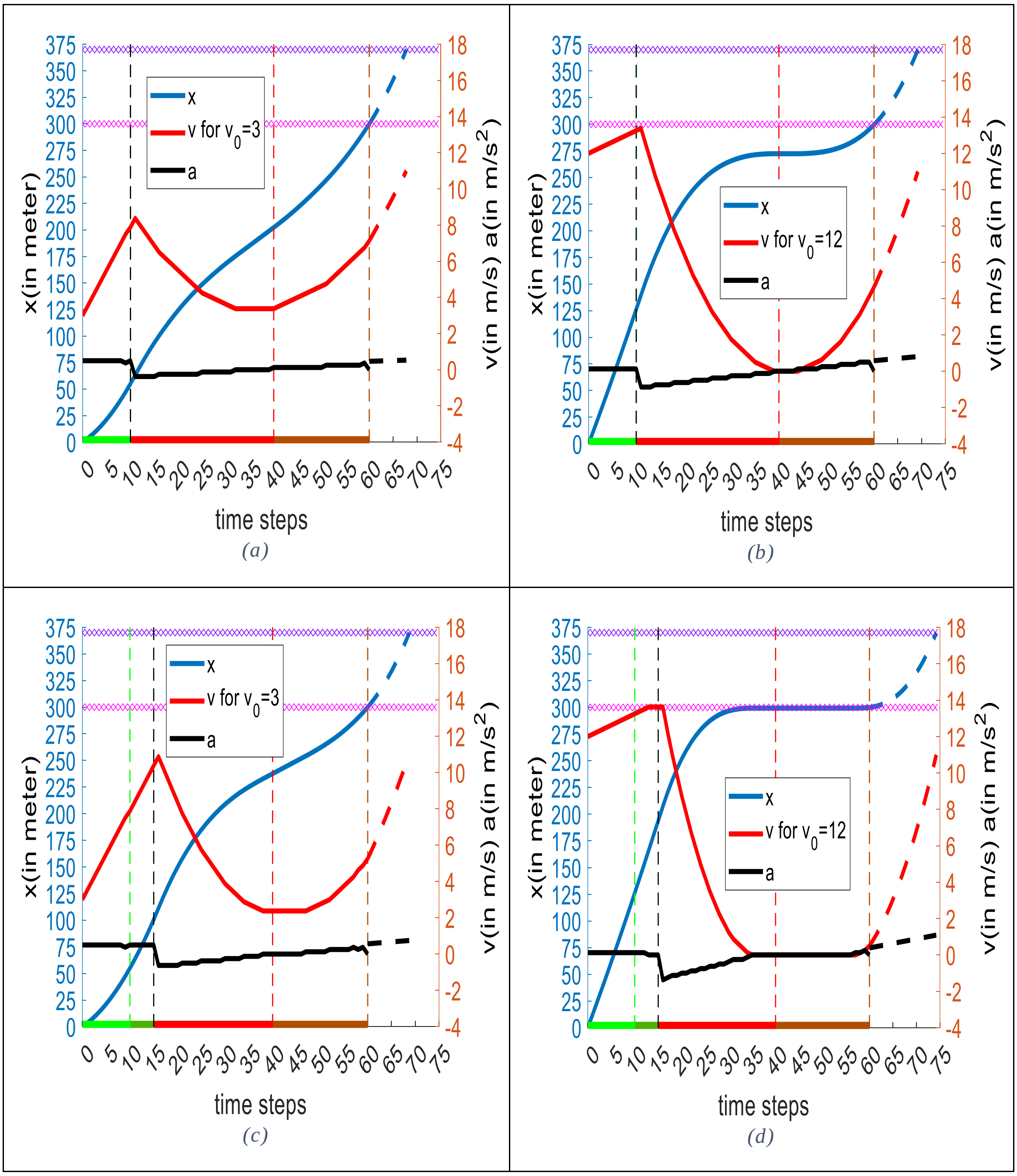

Figure 4. Optimal state and control trajectories of SDP (blue lines) and the continuity of them from the traffic light to the final state (dashed lines). Velocity and control (acceleration) for each trajectory (red and black lines) and their continuity till the final state with dashed lines. The left vertical axis is for the blue lines and the right vertical axis is for the red and black lines. The light green horizontal line represents the certain green phase, the green horizontal line represents the uncertain green phase, the red horizontal line represents the certain red phase and the brown horizontal line represents the uncertain red phase. The vertical back dashed line represents the time of switching from green to red. The pink x-line represents the position of the traffic light and the purple x-line the position of the final state.

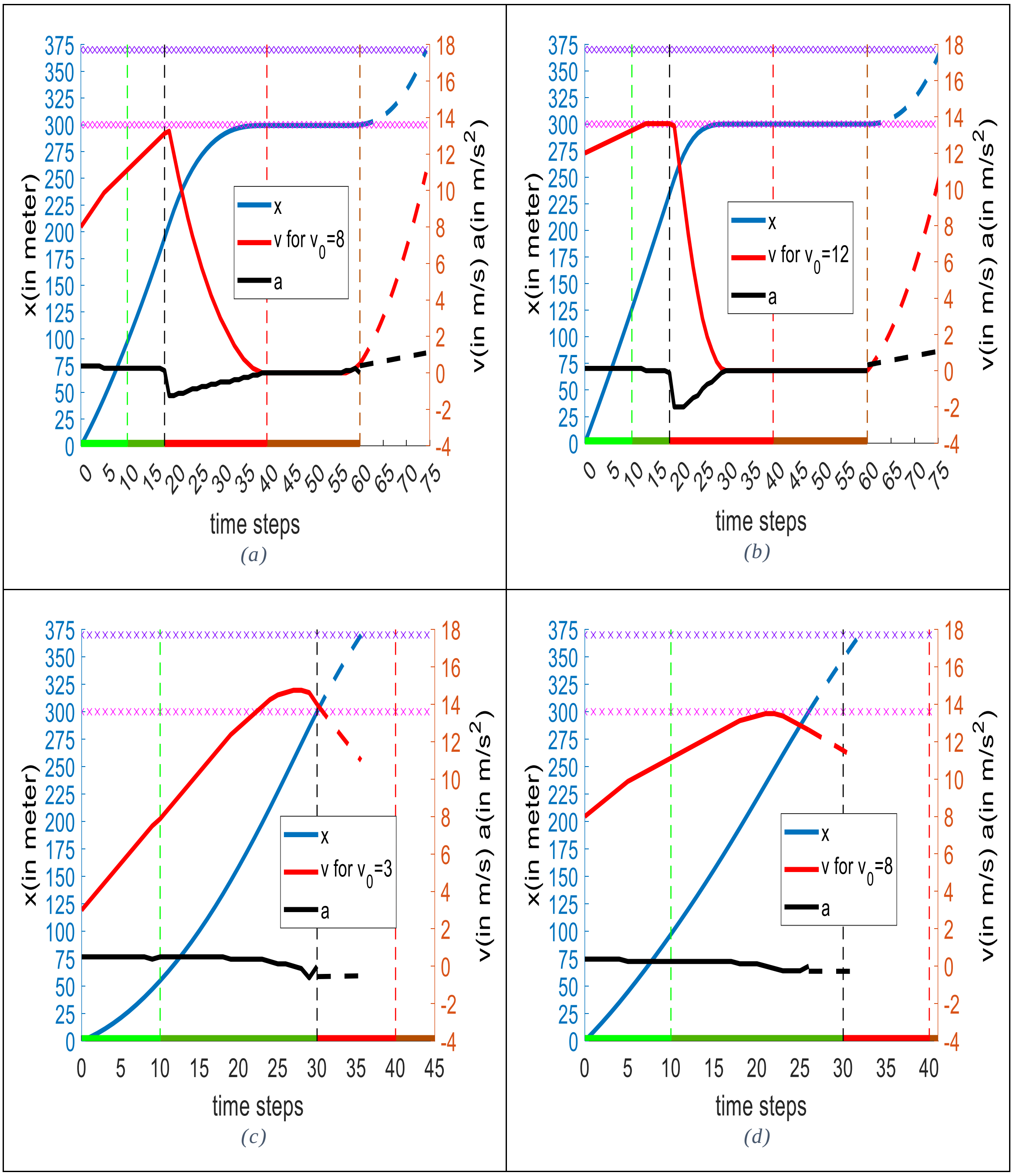

Figure 5. Optimal state and control trajectories of SDP (blue lines) and the continuity of them from the traffic light to the final state (dashed lines). Velocity and control (acceleration) for each trajectory (red and black lines) and their continuity till the final state with dashed lines. The left vertical axis is for the blue lines and the right vertical axis is for the red and black lines. The light green horizontal line represents the certain green phase, the green horizontal line represents the uncertain green phase, the red horizontal line represents the certain red phase and the brown horizontal line represents the uncertain red phase. The vertical back dashed line represents the time of switching from green to red. The pink x-line represents the position of the traffic light and the purple x-line the position of the final state.

3.1.1.2 Traffic light at $\boldsymbol{x_1 = 350m}$

In this next segment, the presented trajectories are for a 350-meter distance of the traffic light from the initial position of the vehicle. More specifically, for actual switching times $Tg_s = 10$s-$v_0 = 3m/s, 8m/s$ and for $Tg_s = 15$s with initial speed $3m/s$, the vehicle clears a similar trajectory. In the beginning, the vehicle accelerates in the certain green phase as it tries to pass the traffic light before the switch from green to red. Later, the vehicle decelerates after the actual switching time in the remaining uncertain green phase and in the start of the certain red phase, because it understands that it will not be able to pass the traffic signal before the switching from green to red and if it keeps gaining velocity it will have to immobilize. The initial speed affects how much the vehicle accelerates in the start and decelerates afterwards. In the last phase (uncertain red phase) the vehicle accelerates once again to reach an appropriate speed to pass the traffic light at the last possible time step.

For $Tg_s = 10$s-$v_0 = 12m/s$,$Tg_s = 15$s-$v_0 = 8m/s, 12m/s$ and $Tg_s = 18$s-$v_0 = 3m/s, 8m/s$ the vehicle accelerates until the switching to red and then it decelerates, since he did not have time to pass. In the end of the certain red phase, it immobilizes and afterwards it accelerates again at the start of the uncertain red phase. In addition, for $Tg_s = 18$s-$v_0 = 12m/s$ the trajectory is similar with the difference that the vehicle stops in a very close distance to the traffic light.

On the other hand, a different type of trajectory emerges for the cases of $Tg_s = 30s$ with initial speeds $3m/s, 8m/s, 12m/s$. In the cases with the above-mentioned specifications the vehicle passes the traffic light before the first switch. The vehicle accelerates the whole certain green phase and continues in the start of the uncertain green phase. Afterwards, the speed remains constant for the rest of the uncertain green phase where the passing of the traffic light occurs.

In the table below are presented the fuel consumption based on the ARRB model and the final time t_e that the vehicle reaches the final position.

$v_0 - Tg_s$	Fuel (ml)	$t_e\ (s)$
3.0m/s-10s	78.5669	65.1109
8.0m/s-10s	77.1048	65.2643
12.0m/s-10s	77.1755	65.7841
3.0m/s-15s	83.2326	65.3730
8.0m/s-15s	80.9846	65.4922
12.0m/s-15s	79.0394	66.4030
3.0m/s-18s	85.9669	65.6078
8.0m/s-18s	81.8963	66.1953
12.0m/s-18s	78.9904	68.6057
3.0m/s-30s	53.7756	34.3658

8.0m/s-30s	57.8863	31.9088
12.0m/s-30s	50.9954	28.3790

Table 4. *Fuel consumption (ARRB model) and final time for a combination of initial velocity and actual switching time.*

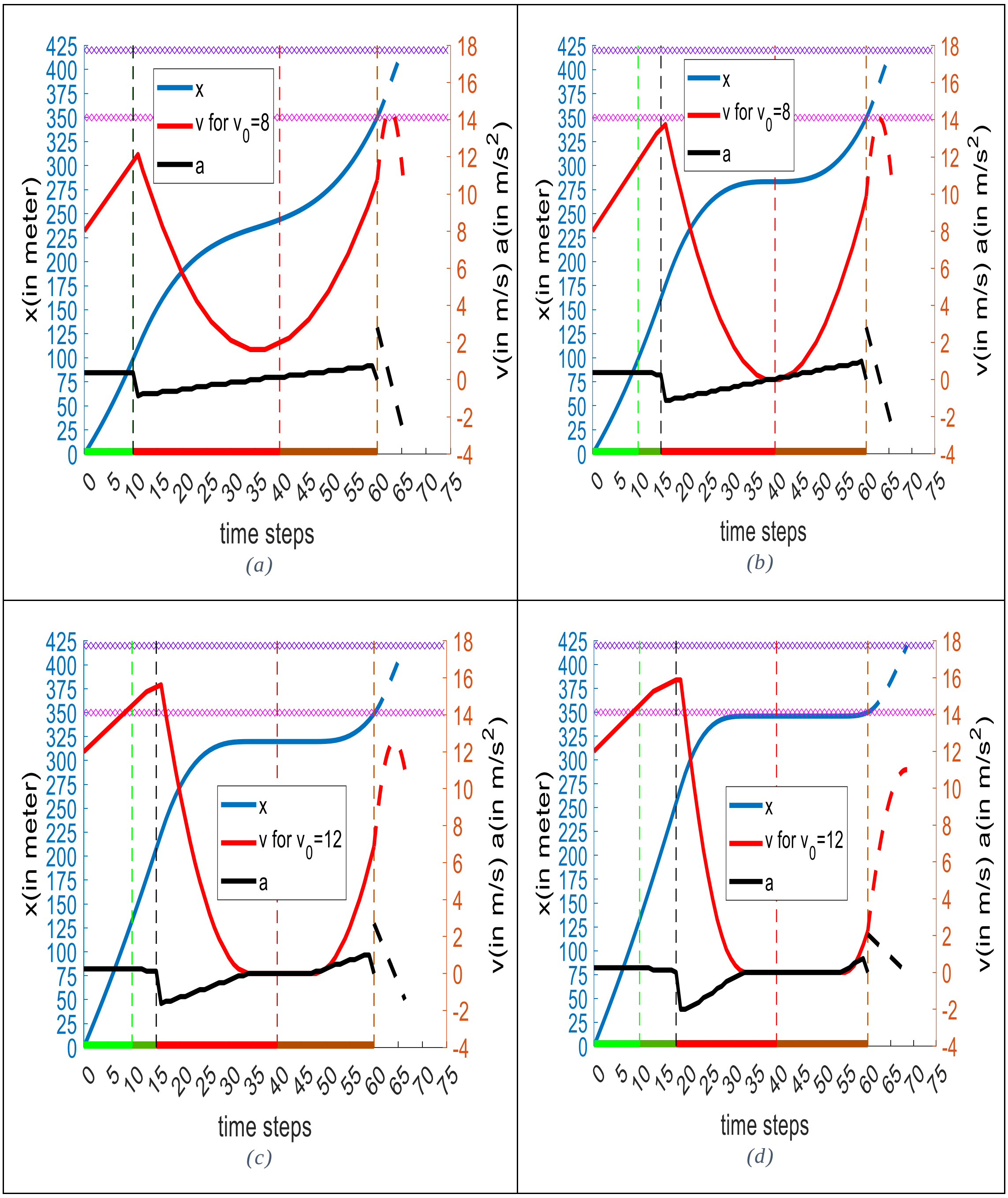

Figure 6. Optimal state and control trajectories of SDP (blue lines) and the continuity of them from the traffic light to the final state (dashed lines). Velocity and control (acceleration) for each trajectory (red and black lines) and their continuity till the final state with dashed lines. The left vertical axis is for the blue lines and the right vertical axis is for the red and black lines. The light green horizontal line represents the certain green phase, the green horizontal line represents the uncertain green phase, the

red horizontal line represents the certain red phase and the brown horizontal line represents the uncertain red phase. The vertical back dashed line represents the time of switching from green to red. The pink x-line represents the position of the traffic light and the purple x-line the position of the final state.

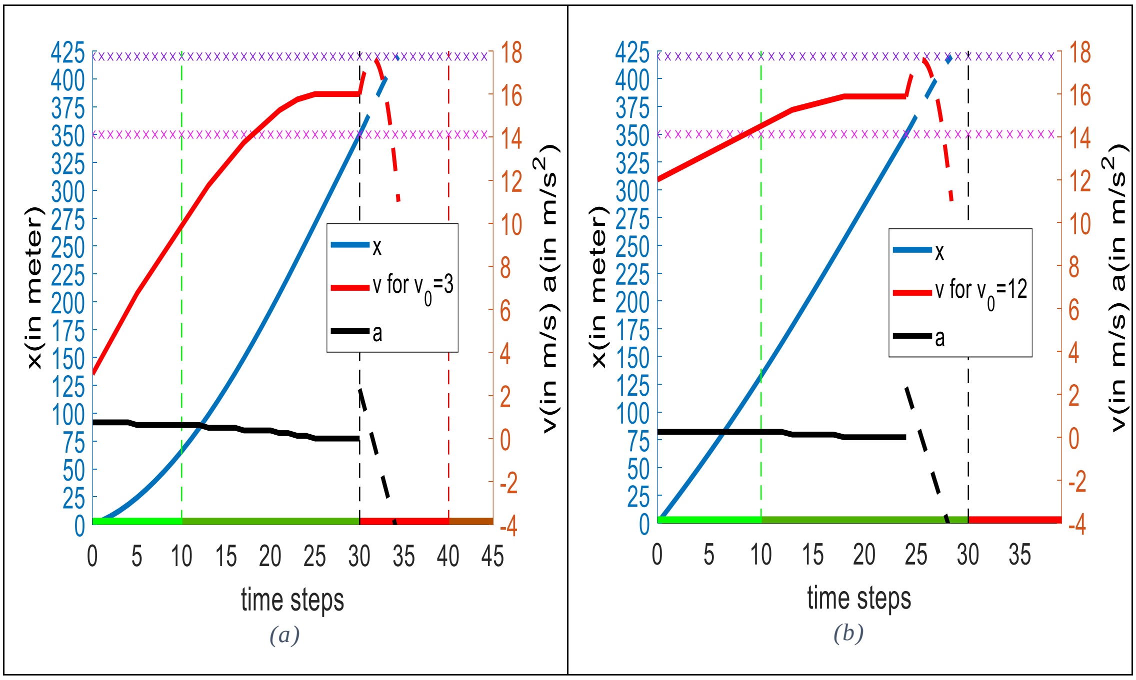

Figure 7. Optimal state and control trajectories of SDP (blue lines) and the continuity of them from the traffic light to the final state (dashed lines). Velocity and control (acceleration) for each trajectory (red and black lines) and their continuity till the final state with dashed lines. The left vertical axis is for the blue lines and the right vertical axis is for the red and black lines. The light green horizontal line represents the certain green phase, the green horizontal line represents the uncertain green phase, the red horizontal line represents the certain red phase and the brown horizontal line represents the uncertain red phase. The vertical back dashed line represents the time of switching from green to red. The pink x-line represents the position of the traffic light and the purple x-line the position of the final state.

3.1.1.3 Traffic light at $\boldsymbol{x_1 = 150m}$

In this last segment of the first scenario are presented the trajectories with the traffic light at the 150 meters. In the cases of $Tg_s = 10s, 15s, 18s$ with $v_0 = 3m/s$ the vehicle gains speed until the switch to red. Then it slows down to his immobilization. At the uncertain red phase, the vehicle accelerates again to pass the traffic light at the moment of the green switching.

For $Tg_s = 10s$ with $v_0 = 8m/s$ the vehicle accelerates until the first switch, then it decelerates until it has reached the 150-mark distance where it immobilizes. After the switch to green it accelerates to meet the final state position and speed.

In the cases of $Tg_s = 15s, 18s, 30s$-$v_0 = 8m/s, 12m/s$ the vehicle passes the traffic light before the switch. That is a result of the combination of Tg_s, v_0 and x_1 . The vehicle does not need a big period of time to reach the 150 meters with the current limits of velocity and acceleration. If the traffic light switches at the first 5 seconds of the uncertain green phase, the vehicle will follow a trajectory similar to the Figure 8a. Regarding the trajectory of this case, the vehicle gains speed throughout his course to reach a velocity between $14m/s - 15m/s$ at the 150-meter mark and afterwards it gradually decelerates to arrive at the final state with the required final velocity. For the case of $Tg_s = 30s$-$v_0 = 3m/s$ the vehicle handles to pass the traffic light at the last time step before the switch to red. Firstly, the vehicle accelerates in the first 15 seconds, then keeps the velocity intact. Afterwards it decelerates because it is at a close distance to the traffic light, so it can immobilize. Finally, the vehicle reaccelerates and passes the traffic light at the last second.

In the table below are presented the fuel consumption based on the ARRB model and the final time t_e that the vehicle reaches the final position (220 meters in this case).

$v_0 - Tg_s$	Fuel (ml)	t_e (s)
3.0m/s-10s	68.6904	65.6037
8.0m/s-10s	70.2847	71.7586
12.0m/s-10s	-	-
3.0m/s-15s	67.9187	66.0173
8.0m/s-15s	38.6604	18.7400
12.0m/s-15s	32.5955	15.7417
3.0m/s-18s	66.9384	66.2780
8.0m/s-18s	38.6604	18.7400
12.0m/s-18s	32.5955	15.7417
3.0m/s-30s	27.3306	35.9218
8.0m/s-30s	38.6604	18.7400
12.0m/s-30s	32.5955	15.7417

Table 5. *Fuel consumption (ARRB model) and final time for a combination of initial velocity and actual switching time.*

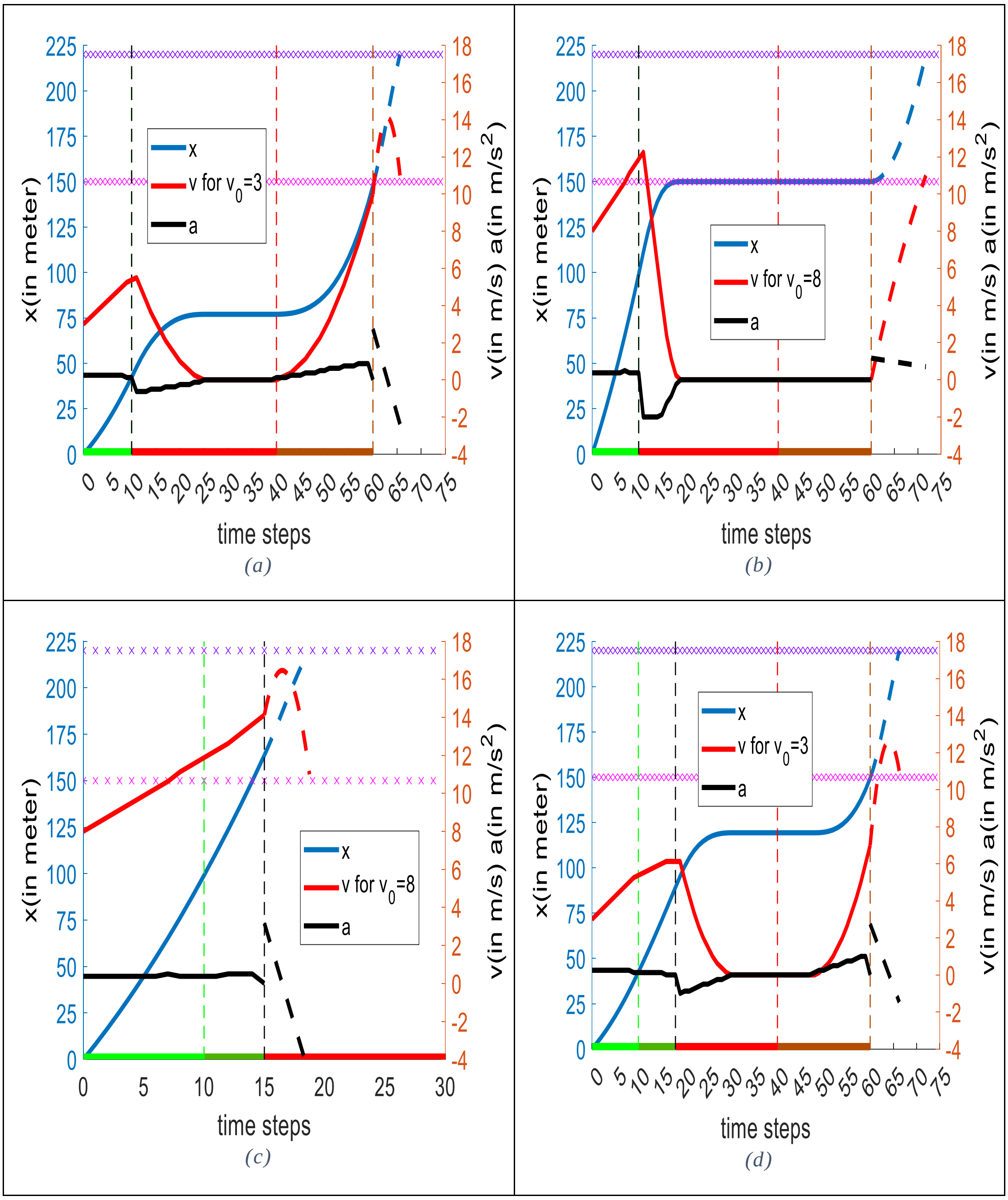

Figure 8. Optimal state and control trajectories of SDP (blue lines) and the continuity of them from the traffic light to the final state (dashed lines). Velocity and control (acceleration) for each trajectory (red and black lines) and their continuity till the final state with dashed lines. The left vertical axis is for the blue lines and the right vertical axis is for the red and black lines. The light green horizontal line represents the certain green phase, the green horizontal line represents the uncertain green phase, the red horizontal line represents the certain red phase and the brown horizontal line represents the uncertain red phase. The vertical back dashed line represents the time of switching from green to red. The pink x-line represents the position of the traffic light and the purple x-line the position of the final state.

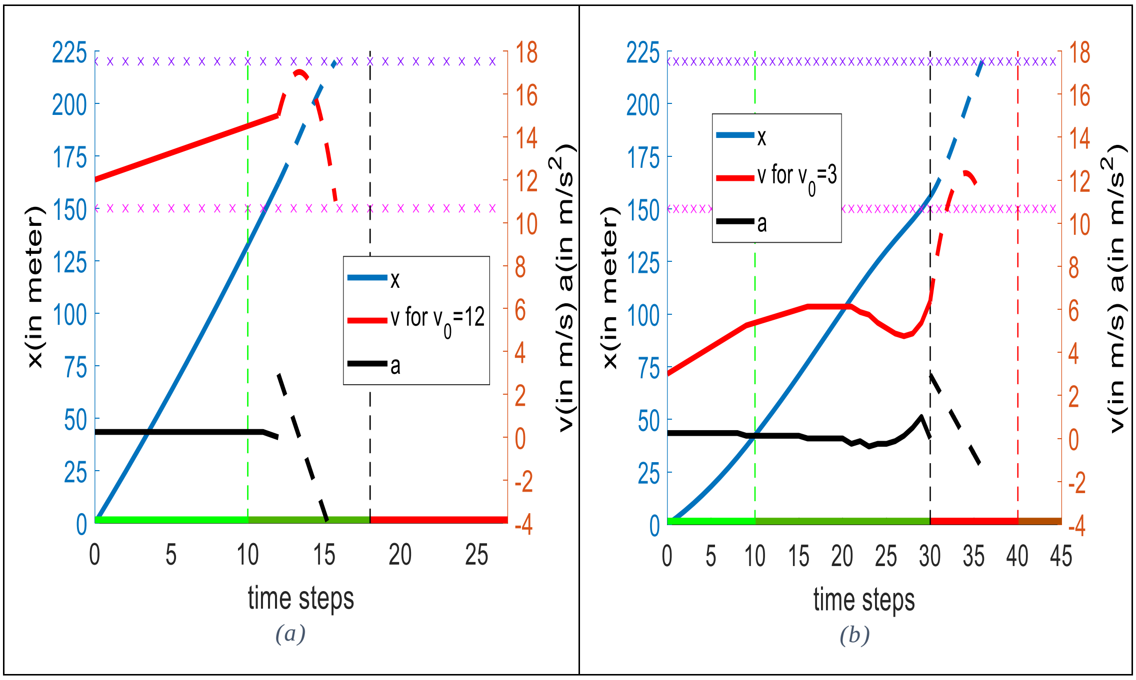

Figure 9. Optimal state and control trajectories of SDP (blue lines) and the continuity of them from the traffic light to the final state (dashed lines). Velocity and control (acceleration) for each trajectory (red and black lines) and their continuity till the final state with dashed lines. The left vertical axis is for the blue lines and the right vertical axis is for the red and black lines. The light green horizontal line represents the certain green phase, the green horizontal line represents the uncertain green phase, the red horizontal line represents the certain red phase and the brown horizontal line represents the uncertain red phase. The vertical back dashed line represents the time of switching from green to red. The pink x-line represents the position of the traffic light and the purple x-line the position of the final state.

3.1.2 Changing actual switching time Tg_s in the range of 10s-30s (**Scenario 2**)

The following diagram shows the trajectory of the vehicle with different Tg_s (actual switching time from green to red). The length of the phases is the same as in the original scenario. The traffic light is set at 300 meters and the initial speed of the vehicle is $v_0 = 12.0m/s$. The vertical lines correspond to the Tg_s and has the same color as the trajectory of the vehicle.

It is clear that the actual switching Tg_s of the traffic light plays a very important role in the trajectory of the vehicle. For values under 23 ($Tg_s < 23$) the vehicle does not have the time to pass the traffic light before the switching. At these cases the trajectory is similar. The only difference is that the later the actual switching time Tg_s, the more covered distance to the traffic light it has during the green phase (certain and uncertain). For values over 22 ($Tg_s \geq 23$) the vehicle passes the 300-meter mark (traffic signal) before the switch from green to red and continues its course to the final state at the 370 meters. For the case of $Tg_s = 23,24,26,28,30$ the trajectories are exactly the same, that explains why the graph has six trajectories for ten different Tg_s. This trajectory occurs because it is the optimal when the vehicle has the time to pass the traffic light.

There is also a big distinction in the consumption of fuel between the two different types of trajectories. More specifically for $Tg_s = 10 - 18$, the

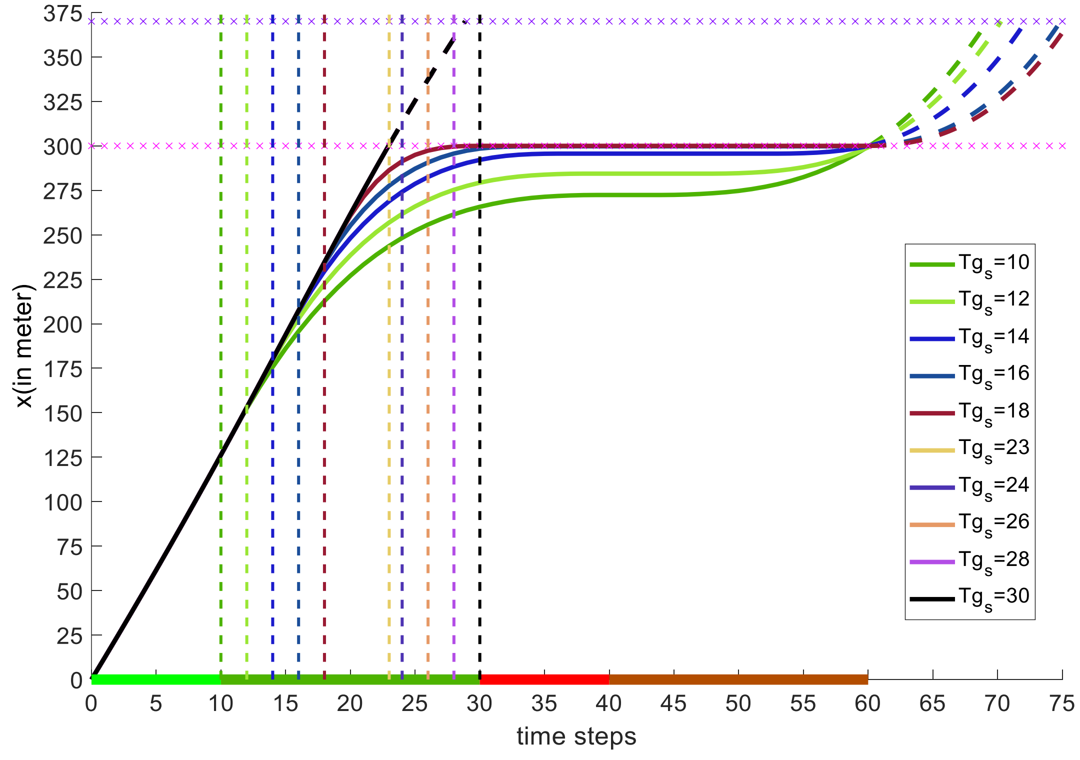

Figure 10*. Trajectories of consecutive values of the actual switching time from 10s to 30s with initial velocity 12m/s.*

required fuel based on the ARRB model (Akcelik et al. 1987) to reach the final state is from 66.7992ml to 76.1527ml (Table 6). The consumption of fuel is smaller for quicker actual switching times, because the vehicle does not accelerate that much before its immobilization. In the other case ($Tg_s = 23 - 30$) the required fuel is 35.1281ml (47.41%-53.87% less fuel in the case that the vehicle passes the traffic light before the actual switching Tg_s).

Tg_s	Fuel (ml)	$t_e\ (s)$
10	66.7992	69.2255
12	69.0545	70.2722
14	71.5894	72.1705
16	74.641	74.8656
18	76.1527	75.5951
23,24,26,28,30	35.1281	28.8663

Table 6. *Fuel consumption (ARRB model) and final time for initial velocity 12m/s and changing actual switching time.*

3.1.3 General outcomes regarding the scenarios

Outcomes based on the specific separation of the phases ($T^r_{max} = 60.0, T^r_{min} = 40.0, T^g_{max} = 30.0, T^g_{min} = 10.0$) and the position of the traffic light x_1 at 150m, 300m, 350m.

- The SDP algorithm relies in an a-priori "crop-and-scale" probability $\mathrm{P}(k^R_{min})$ in the stochastic phases as shown in Equation 26. This uncertainty of the switching leads the vehicle to accelerate and attempt to pass the traffic light before the first switch (from green to red).
- When the vehicle gets to pass the traffic light before the switch from green to red, the consumption of fuel is much better based on the ARRB model. This can be justified by the close to linear trajectory and in the lack of fluctuations in the speed.
- The vehicle reaches the final state much faster when it passes the traffic light before the switch from green to red.
- The ratio of the initial speed to the position of the traffic light and in relation to the actual switching time plays an important role to the trajectory of the vehicle.

3.2 Conclusion

In earlier work a stochastic GLOSA methodology was developed (Typaldos et al., 2020) where through the implementation of SDP techniques, optimal kinematic trajectories of a vehicle are generated, considering a stochastic traffic signal switching, with fixed final state and free final time. In continuation of this work that regards only half traffic light cycle (initially is red and switches to green), an extension has been developed considering a full traffic light cycle. The cycle consists of four phases, a certain green phase, an uncertain green phase, a certain red phase and an uncertain red phase. Results of the extended GLOSA gives us information about how initial velocity, actual switching time, position of the traffic light and the temporal separation of the four phases influence the vehicle's trajectories.

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