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Spatio-temporal Estimation in Wireless Networks with Message Passing

by

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Abstract

In wireless networks, position information is a vital requirement for the network to function as intended. Localization is usually based on time of arrival (TOA) measurements, and thus, accurate timing information is essential for time-based localization. In this thesis, we utilize timing and ranging information exchange between neighboring nodes to overcome this problem. Initially, we approach the problem of spatio-temporal estimation in a sequential manner. We first aim to synchronize the nodes, using prior art time synchronization methods based on linear programming or Gaussian belief propagation and then estimate their locations, using (cooperative) Gaussian belief propagation. Afterwards, we attempt to solve the problem in a joint way, by estimating the clock offsets and the locations simultaneously, assuming that the nodes exchange messages with the anchors (non-cooperatively). In both methods, we focus on proposing efficient belief propagation (BP)-based algorithms that reduce communication overhead and computational complexity. Towards this goal, we approximately linearize the nonlinear terms of the factor graph (FG) messages in order to obtain a closed-form Gaussian solution of message updates. Accordingly, only the means and variances need to be updated and transmitted by the network nodes. Finally, we present the numerical results of each method and discuss their advantages and drawbacks. Considering time offsets of order $\sim 10^{-8}$ sec (10 nanosec) and a $50 \times 50\,\mathrm{m}^2$ plane, with 50 agents and 9 anchors, simulations showed that the proposed cooperative GBP can provide quite accurate location estimates, just 0.8 m off from the real values, under 20 m communication range and a noise variance for ranging measurements of $\sigma_d^2 = 1\,\mathrm{m}^2$, utilizing the sequential method. Under the same circumstances, joint estimation in a non-cooperative environment demonstrates a slightly larger average error, ~ 1 m, but lowers the communication requirements of the network. In the joint non-cooperative case, the number of the messages that need to be exchanged is significantly reduced due to the lower number of neighbors per agent, since a circle of $20\,\mathrm{m}$ radius in a $50 \times 50\,\mathrm{m}^2$ plane would contain many more agents than anchors, $15 - 20$ and $5 - 7$ on average, respectively.

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Chapter 1

Introduction

Wireless networks play a significant role in the modern societies. For many applications of wireless networks, such as public service, emergence rescue and intelligent vehicular system, position information is a crucial requirement for the network to function as intended. Generally, the Global Positioning System (GPS) can provide accurate location in our daily life. However, equipping all wireless nodes (e.g., sensors, vehicles, people) with GPS receivers may be expensive and energy prohibitive. Furthermore, the poor signal penetration capabilities of the widely-used GPS lead to inadequate location information.

A cooperative localization algorithm [1] that enables ranging and position information exchange between neighboring nodes can overcome this problem. This algorithm relies on a set of nodes with known locations, also known as anchors, and on a set of range measurements among neighboring nodes in order to perform localization of the agent nodes, i.e., nodes whose location is currently unknown. The range measurements can be obtained using time of arrival (TOA), time difference of arrival (TDOA), round-trip time of arrival (RTT) or received signal strength (RSS) measurements. RSS measurements have the drawback of being sensitive to changes in the environment, whereas the time-based methods alleviate this problem. Moreover, since TDOA and RTT mechanisms may not be supported by many communication protocols, we will focus on TOA-based techniques.

However, utilizing TOA measurements to obtain accurate range estimates is difficult in the presence of time offsets among the nodes. Hence, clock synchronization is a vital requirement in TOA-based localization methods. A factor graph (FG) based distributed network synchronization algorithm using Belief Propagation (BP) is proposed in [2]. This work requires round-trip time of arrival in order to overcome the unknown positions of the agent nodes, and exploits Gaussian Belief Propagation (GBP) to synchronize the agents of the network.

Having synchronized the nodes to the reference time, we are then able to estimate their locations using the cooperative sum product algorithm, which is developed in [1] for wireless networks, denoted as SPAWN. However, the range measurements that are necessary to perform localization introduce a non-linear term in the observation function, which makes SPAWN a quite expensive algorithm due to the high communication overhead it requires. In this work, we use Taylor expansions that have been proposed in [3] to linearize this term, and, thus compute all the messages on the FG in closed Gaussian form. This means that we will only be dealing with means and variances of the node beliefs, which can be easily obtained by addition and multiplication operations. The complexity and communication requirements of this algorithm are much lower than those of sample-based algorithms, such as SPAWN [1].

It is obvious that this entire process treats localization independently of the time synchronization task. However, the two problems are closely related and it is possible to explore a joint estimation method. Furthermore, in a harsh or mobile environment, the clock of the nodes varies and re-synchronization between nodes frequently increases the energy consumption. So, we also try to perform spatio-temporal estimation in a joint way. Again, utilizing the Taylor expansions [3] to avoid dealing with non-linear terms, we describe a non-cooperative joint GBP algorithm, which means that the agent nodes can only exchange messages with the anchor nodes. The assumption of a non-cooperative environment significantly reduces the number of neighbors per agent, since there are just a few anchors in the network, and, thus, lowers communication requirements, such as bandwidth. The corresponding cooperative joint algorithm [3], where agents can communicate with each other is also presented.

Finally, in realistic wireless networks, the clock readings of the agent nodes are affected not only by the time offset, but by a frequency offset as well, that also needs to be estimated in order to entirely synchronize the agent nodes to the reference time of the network. So, we extend our reading model to include both the time and frequency offsets, and we use the Linear Programming method that was proposed in [4], where it is assumed that synchronization take place in a non-cooperative environment. We then exploit the proposed GBP based on the Taylor approximation to locate the agents, and carry out successfully the spatio-temporal estimation.

The rest of the thesis is organized as follows. In Chapter 2, we present the Setup and System Model, we are based on throughout this work. We study the case of Sequential Spatio-temporal Estimation in Chapter 3: Cooperative Synchronization is presented in Section 3.1, Non-cooperative Synchronization in Section 3.2, and Cooperative Localization in Section 3.3. Moving on to Chapter 4, we talk about Joint Spatio-temporal Estimation, and separate our discussion on Joint Non-cooperative Estimation in Section 4.1 and Joint Cooperative Estimation in Section 4.2. Finally, we present the Numerical Results in Chapter 5, and draw some Conclusions in Chapter 6. There is also an Appendix (Chapter 7), where we derive the basic equations of the thesis.

Chapter 2

Setup & System Model

We consider a wireless network with $\mathcal{M} = \{1, \dots, M\}$ the set of agent nodes to be located and time-synchronized and $\mathcal{A} = \{1, \dots, A\}$ the set of anchor nodes with known coordinates, assumed to be time-synchronized at the same reference time. The local clock of each node i is modelled as follows:

$$c_i(t) = \phi_i \, t + \theta_i \, , \tag{2.1}$$

where t is the accurate reference time, θ_i is the time offset and ϕ_i is the frequency offset of node i. If $i \in \mathcal{A}$, then $\theta_i = 0$ and $\phi_i = 1$. If node i and node j are within reliable communication range, then these two nodes are considered to be neighbors. The set of all available communication links is denoted as Ξ, and the set of all neighbors of node i as $N(i)$.

As shown in Fig. 2.1, node i transmits its current timing information to node j, at real time t_1, with its clock reading $c_i(t_1)$ embedded in the message. After a delay Δ_{ij}, node j receives the timing information from node i at time t_2 and records its corresponding clock reading $c_j(t_2)$. Node j then sends back a signal at time t_3, which contains both $c_j(t_2)$ and $c_j(t_3)$, and node i captures the backward signal after a delay Δ_{ji}, at time t_4 and records it as $c_i(t_4)$. This process can be repeated for N rounds of message exchange, so after the n-th round, we have collected a set of time stamps $\{c_i(t_1^n), c_j(t_2^n), c_j(t_3^n), c_i(t_4^n)\}_{n=1}^N$.

The delay Δ_{ij} is the signal propagating time $\frac{d_{ij}}{c}$, with the Euclidean distance $d_{ij} = ||\mathbf{x}_j - \mathbf{x}_i||$ and the speed of light c. Thus $\Delta_{ij} = \Delta_{ji}$. The measured time stamps at the receivers are modelled as follows:

$$c_j(t_2^n) = \phi_j \left(\frac{c_i(t_1^n) - \theta_i}{\phi_i} + \frac{d_{ij}}{c} + u_{ij,n} \right) + \theta_j \, , \tag{2.2}$$

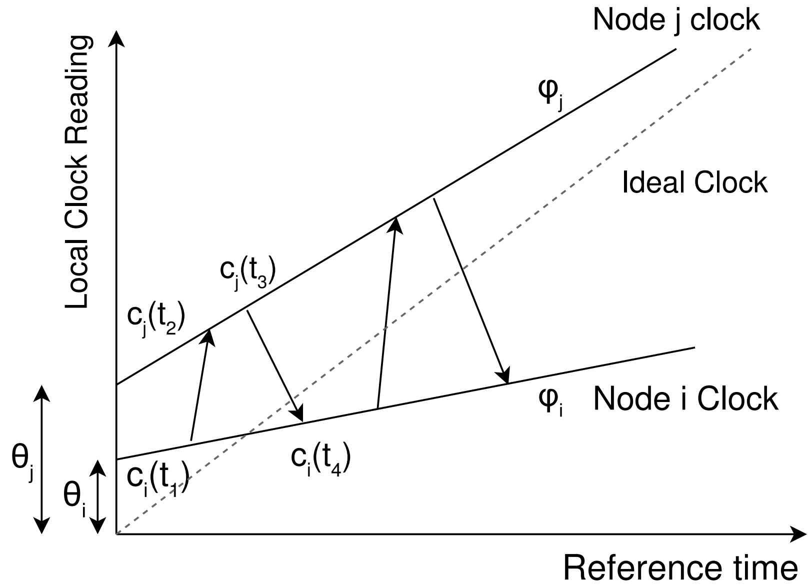

Figure 2.1: Synchronization messages between Node i and Node j in the two-way technique.

$$c_i(t_4^n) = \phi_i\left(\frac{c_j(t_3^n) - \theta_j}{\phi_j} + \frac{d_{ij}}{c} + u_{ji,n}\right) + \theta_i\,, \tag{2.3}$$

where $\{u_{ij,n}\}_{n=1}^{N}$ and $\{u_{ji,n}\}_{n=1}^{N}$ are assumed to be i.i.d. Gaussian distributed random variables, that play the role of the measurement noise.

If the frequency skew is 0, and, thus $\phi = 1$, the observed signal propagation time can be obtained from timestamps $c(t)$ as follows:

$$t_{ij,n} = c_j(t_2^n) - c_i(t_1^n) = \frac{||\mathbf{x}_j - \mathbf{x}_i||}{c} + (\theta_j - \theta_i) + u_{ij,n}\,, \tag{2.4}$$

where $u_{ij,n} \sim \mathcal{N}(0, \sigma_t^2)$, for $n = 1, \ldots, N$. Multiplying both sides of Eq. (2.4) by c, we have

$$z_{ij,n} = ||\mathbf{x}_j - \mathbf{x}_i|| + c(\theta_j - \theta_i) + \zeta_{ij,n}\,, \tag{2.5}$$

where $\zeta_{ij,n} = c \cdot u_{ij,N}$ is also Gaussian distributed, i.e., $\zeta_{ij,N} \sim \mathcal{N}(0, \sigma_d^2)$, with $\sigma_d^2 \triangleq c^2\sigma_t^2$, for $n = 1, \ldots, N$. It is noted that $\zeta_{ij,n}$ is assumed independent of $\zeta_{i'j',n}$ for any $i' \neq i$ or $j' \neq j$, and for $n = 1, \ldots, N$.

Chapter 3

Sequential Spatio-temporal Estimation

This Chapter describes how we can synchronize and locate the agent nodes of the wireless network in a sequential manner. More specifically, we firstly synchronize the agents by estimating their clock parameters, θ and ϕ, and then we perform localization. For both problems, we map the network into a factor graph (FG) and infer the required parameters, i.e., clock & frequency offset and location, using Belief Propagation-based algorithms.

3.1 Cooperative Synchronization

In this Section, we aim to estimate the clock offsets, θ, of the agent nodes of the wireless network, considering that there is no frequency skew in the measurement model, i.e., the frequency offset is equal to 1. The analysis presented below is based on [2].

As described in Chapter 2, at the n-th round of information packet exchange between node i and node j, the measured time stamps at the receivers, for $\phi = 1$, are modelled as:

$$c_j(t_2^n) - \theta_j = c_i(t_1^n) - \theta_i + \frac{d_{ij}}{c} + u_{ij,n}\,, \tag{3.1}$$

$$c_j(t_3^n) - \theta_j = c_i(t_4^n) - \theta_i + \frac{d_{ij}}{c} - u_{ji,n}\,. \tag{3.2}$$

By adding (3.1) to (3.2) and denoting $c_j(t_2^n)+c_j(t_3^n)-c_i(t_1^n)-c_i(t_4^n)$ as $T_{\{i,j\},n}$, we obtain:

$$T_{\{i,j\},n} = 2(\theta_j - \theta_i) + w_n, \quad \text{for } n = 1, \ldots, N, \tag{3.3}$$

where $\{w_n\}_{n=1}^N$ are i.i.d. Gaussian random variables with zero mean and variance σ^2.

If we stack the observations $\left\{T_{\{i,j\},n}\right\}_{n=1}^N$ in a vector, as

$$\mathbf{T}_{\{i,j\}} = \left[T_{\{i,j\},1}, \ldots, T_{\{i,j\},N}\right], \tag{3.4}$$

we have the following likelihood function:

$$\begin{aligned} p(\mathbf{T}_{i,j} \,|\, \theta_i, \theta_j) &= \frac{1}{(\sqrt{2\pi}\sigma)^N} \exp\left\{ -\frac{1}{2\sigma^2} ||\mathbf{T}_{i,j} - 2(\theta_j - \theta_i)\mathbf{1}||^2 \right\} \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^N} \exp\left\{ -\frac{1}{2}\left(\frac{\sigma^2}{N}\right)^{-1} \left[2(\theta_i - \theta_j) + \frac{1}{N}\mathbf{1}^T\mathbf{T}_{i,j}\right]^2 \right\}, \end{aligned} \tag{3.5}$$

where $\mathbf{1}$ is a vector with length N and all its elements equal to 1. By the Bayesian rule, the joint posterior distribution $p(\theta_i, \theta_j|\mathbf{T}_{i,j})$ can be expressed as:

$$p(\theta_i, \theta_j \,|\, \mathbf{T}_{i,j}) \propto p(\mathbf{T}_{i,j} \,|\, \theta_i, \theta_j) p(\theta_i) p(\theta_j), \tag{3.6}$$

with $p(\theta_i)$ and $p(\theta_j)$ denoting the prior distributions of θ_i and θ_j, respectively. To obtain the estimates of the agents' clock offsets, we first need the marginal posterior distribution of θ_i, $\forall\, i \in \mathcal{M}$. Mathematically

$$\begin{aligned} p(\theta_i \,|\, \mathbf{T}_{i,j}) &= \int p(\theta_i, \theta_j \,|\, \mathbf{T}_{i,j})\, d\theta_j \\ &\propto \int p(\mathbf{T}_{i,j} \,|\, \theta_i, \theta_j) p(\theta_i) p(\theta_j)\, d\theta_j, \end{aligned} \tag{3.7}$$

Now, θ_i can be estimated by maximizing $p(\theta_i \,|\, \mathbf{T}_{i,j})$, which gives us the optimal solution in a Bayesian sense.

If we extend the above idea to a wireless network with M agent nodes,

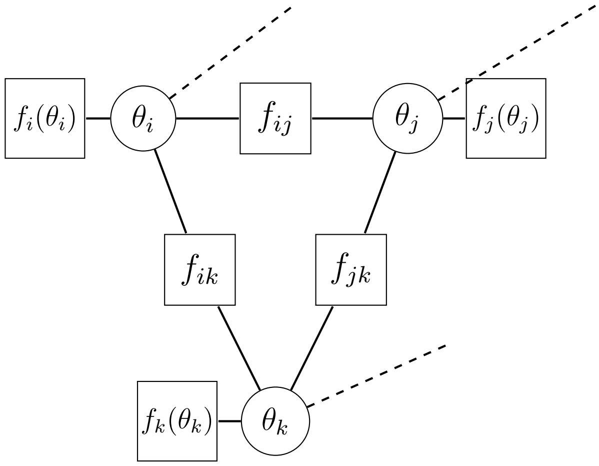

Figure 3.1: Factor graph for message exchange between Node i, Node j and Node k for synchronization, with $f_{ij} = p(\mathbf{T}_{i,j} \,|\, \theta_i, \theta_j)$ and $f(\theta)$ the prior of each variable. The dashed line denotes that the variable nodes have more neighbors.

the joint posterior distribution becomes

$$p\left(\theta_1, \ldots, \theta_M \,|\, \{\mathbf{T}_{i,j}\}_{i=1,\ldots,M,\, j\in N(i)}\right), \tag{3.8}$$

so the corresponding marginal posterior distribution is

$$p\left(\theta_i \,|\, \{\mathbf{T}_{i,j}\}_{i=1,\ldots,M,\, j\in N(i)}\right) = \int \cdots \int p\left(\theta_1, \ldots, \theta_M \,|\, \{\mathbf{T}_{i,j}\}_{i=1,\ldots,M,\, j\in N(i)}\right) d\theta_1 \ldots d\theta_{i-1}\, d\theta_{i+1} \ldots d\theta_M, \tag{3.9}$$

An efficient way to calculate the marginal distributions of every variable $\theta_i, \forall\, i \in \mathcal{M}$ is to create the factor graph of the joint posterior distribution and run the Belief Propagation algorithm on it. For example, if three agent nodes can communicate with each other, and, thus, are considered to be neighbors, the corresponding factor graph of their clock offset variables, $\theta_i, \theta_j, \theta_k$, would be the one in Fig. 3.1. In this problem, $f_{ij} = f_{ji}$, $\forall\, (i, j) \in \Xi$.

Belief Propagation involves two kind of messages, at each iteration l:

- $b_i(\theta_i)$, which is the belief of variable node θ_i and is defined as the product of all incoming messages from the neighboring factor nodes:

$$b_i^{(l)}(\theta_i) = m_{f_i \to \theta_i}^{(l)} \prod_{j \in \mathcal{N}(i)} m_{f_{ij} \to \theta_i}^{(l-1)}(\theta_i)\,, \tag{3.10}$$

- $m_{f_{ij} \to \theta_i}^{(l)}(\theta_i)$ which can be calclulated as:

$$m_{f_{ij} \to \theta_i}^{(l)}(\theta_i) = \int p(\mathbf{T}_{i,j} \,|\, \theta_i, \theta_j) b_j^{(l)}(\theta_j)\, d\theta_j\,, \tag{3.11}$$

Notice that for the message $m_{f_i \to \theta_i}^{(l)}$, we have:

$$\begin{aligned} m_{f_i \to \theta_i}^{(l)} &= p(\theta_i) \int b_j^{(l)}(\theta_j) \\ &= p(\theta_i)\,, \end{aligned} \tag{3.12}$$

since $b_j^{(l)}(\theta_j)$ is a PDF and, thus, its integral is equal to 1.

Now, we need to obtain the aforementioned messages in closed form. Let us focus on the computation of the beliefs at first. If node i is an anchor, meaning that it has a known location and is synchronized to the reference time t, its clock offset is equal to 0. So, its prior distribution can be expressed as the Dirac delta function, i.e., $\delta(\theta_i)$, which is also its belief throughout the process of the algorithm, and it is constant. The Dirac delta function can be expressed as a Gaussian PDF with zero mean and zero variance, so $p(\theta_i) = \delta(\theta_i) = \mathcal{N}(\theta_i; 0, 0)$. On the other hand, if node i belongs to the set of the agents, it is considered that its prior distribution is Gaussian with zero mean and variance $+\infty$, i.e., $p(\theta_i) \sim \mathcal{N}(0, +\infty)$, which is its belief at the iteration $l = 0$, i.e., $b_i^{(0)}(\theta_i) \sim \mathcal{N}(0, +\infty)$. The variance is considered to be $+\infty$, because, in practice, it is very difficult to have prior knowledge about the clock offsets.

At the first iteration, the message $m_{f_{ij} \to \theta_i}^{(0)}$, with j being an anchor node, i.e., $\theta_j = 0$, is

$$\begin{aligned} m^{(0)}_{f_{ij}\to\theta_i}(\theta_i) &= \int p(\mathbf{T}_{i,j}\,|\,\theta_i,\theta_j)\delta(\theta_j)d\theta_j \\ &= p(\mathbf{T}_{i,0}\,|\,\theta_i, 0) \\ &\propto \exp\left\{ -\frac{1}{2}\left(\frac{\sigma^2}{N}\right)^{-1}\left[2\theta_i + \frac{1}{N}\mathbf{1}^T\mathbf{T}_{i,0}\right]^2\right\}. \end{aligned} \tag{3.13}$$

It is clear that the message of (3.13) is in Gaussian form with mean $\nu^{(0)}_{0\to i} = -\frac{1}{2N}\mathbf{1}^T\mathbf{T}_{i,0}$ and variance $C^{(0)}_{0\to i} = \frac{\sigma^2}{4N}$, i.e.,

$$m^{(0)}_{f_{ij}\to\theta_i}(\theta_i) \propto \mathcal{N}(\theta_i; \nu^{(0)}_{0\to i}, C^{(0)}_{0\to i}) \,. \tag{3.14}$$

If node j is an agent node, the message $m^{(0)}_{f_{ij}\to\theta_i}$ is:

$$\begin{aligned} m^{(0)}_{f_{ij}\to\theta_i}(\theta_i) &= \int p(\mathbf{T}_{i,j}\,|\,\theta_i,\theta_j) b^{(0)}_j(\theta_j)d\theta_j \\ &\overset{(*)}{\propto} \int p(\mathbf{T}_{i,j}\,|\,\theta_i,\theta_j)d\theta_j \\ &\overset{(**)}{\propto} 1\,, \end{aligned} \tag{3.15}$$

where in $(*)$, we used $b^{(0)}_j(\theta_j) \sim \mathcal{N}(\theta_j; 0, +\infty)$, and in $(**)$ is due to the fact that the integral is independent from θ_i. So, based on (3.10), (3.13) and (3.15), we can write the updated belief $b^{(0)}_i(\theta_i)$ as a Gaussian PDF:

$$b^{(1)}_i(\theta_i) \sim \mathcal{N}\left(\theta_i; \mu^{(1)}_i, P^{(1)}_i\right)\,, \tag{3.16}$$

where $\mu^{(1)}_i = \nu^{(0)}_{0\to i}$ and $P^{(1)}_i = C^{(0)}_{0\to i}$ if node i is directly connected to an achor node, and $\mu^{(1)}_i = 0$ and $P^{(1)}_i = +\infty$ otherwise.

Moving on to the next iteration, variable nodes broadcast their new beliefs to neighboring factor nodes and this results in new messages at the intermediate factor nodes. Specifically, with the belief $b^{(1)}_i(\theta_i)$ at θ_i, the new message from f_{ki} to a neighboring node θ_k, $m^{(1)}_{f_{ki}\to\theta_k}(\theta_k)$, can be obtained as

$$\begin{aligned} m^{(1)}_{f_{ki}\to\theta_k}(\theta_k) &= \int p\left(\mathbf{T}_{k,i} \mid \theta_k, \theta_i\right) b^{(1)}_i(\theta_i)\,\mathrm{d}\theta_i \\ &\propto \int \exp\left\{-\frac{1}{2}\left(\frac{\sigma^2}{N}\right)^{-1}\left(2\theta_k - 2\theta_i + \frac{1}{N}1^T\mathbf{T}_{k,i}\right)^2\right\} \\ &\qquad \times \exp\left\{-\frac{1}{2}\left[P^{(1)}_i\right]^{-1}\left[\theta_i - \mu^{(1)}_i\right]^2\right\}\mathrm{d}\theta_i \\ &\propto \exp\left\{-\frac{1}{2}\left[\frac{\sigma^2}{4N} + P^{(1)}_i\right]^{-1}\left[\theta_k - \mu^{(1)}_i + \frac{1}{2N}1^T\mathbf{T}_{k,i}\right]^2\right\}. \end{aligned} \tag{3.17}$$

It can be seen that the message in (3.17) is also in Gaussian form. By denoting its mean as $\nu^{(1)}_{i\to k} = \mu^{(1)}_i - \frac{1}{2N}\mathbf{1}^T\mathbf{T}_{k,i}$ and variance as $C^{(1)}_{i\to k} = \frac{\sigma^2}{4N} + P^{(1)}_i$, we have

$$m^{(1)}_{f_{ki}\to\theta_k}(\theta_k) \sim \mathcal{N}\left(\theta_k; \nu^{(1)}_{i\to k}, C^{(1)}_{i\to k}\right). \tag{3.18}$$

After collecting all the incoming messages from neighboring factor nodes, variable node θ_k updates its belief $b^{(1)}_k(\theta_k)$ by using (3.10). Since both the prior distribution and the incoming messages are in Gaussian form, the updated belief of θ_k must be Gaussian distributed.

In general, it can be seen that all the messages and beliefs during the iterative procedure are in Gaussian forms. Denoting

$$b^{(l)}_j(\theta_j) \sim \mathcal{N}\left(\theta_j; \mu^{(l)}_j, P^{(l)}_j\right), \forall j \in \{1, \ldots, M\}, \tag{3.19}$$

and with similar calculations in (3.17), we can obtain the message

$$m^{(l)}_{f_{ij}\to\theta_i}(\theta_i) \sim \mathcal{N}\left(\theta_i; \nu^{(l)}_{j\to i}, C^{(l)}_{j\to i}\right), \tag{3.20}$$

$$C^{(l)}_{j\to i} = \frac{\sigma^2}{4N} + P^{(l)}_j, \tag{3.21}$$

and

$$\nu_{j\to i}^{(l)} = \mu_j^{(l)} - \frac{1}{2N}\mathbf{1}^T\mathbf{T}_{i,j}, \tag{3.22}$$

The message in (3.20) represents the general form for all the messages circulating around the network. Recall that the belief at of an anchor node j, i.e., $\delta(\theta_j)$, can be equally represented in the Gaussian form with $P_0^{(I)} = 0$ and $\mu_0^{(l)} = 0$. Putting this into (3.21) and (3.22), it can be seen that (3.20) reduces to (3.13) and it represents the message from the anchor node j to its neighboring sensors. Moreover, for other nodes, their beliefs are initially in Gaussian forms with variance $+\infty$, hence (3.20) reduces to a constant and is consistent with (3.15).

After collecting incoming messages from neighboring factor nodes, the belief of θ_i is updated by using (3.10), that is

$$b_i^{(l+1)}(\theta_i) = p(\theta_i) \prod_{j\in\mathcal{N}(i)} m_{f_{ij}^{(l)}\to\theta_i}(\theta_i), \tag{3.23}$$

Substituting $p(\theta_i) \sim \mathcal{N}(\theta_i; 0, +\infty)$ and (3.20), into (3.23), the variance of $b_i^{(l+1)}(\theta_i)$ is obtained as

$$\begin{aligned}\left[P_i^{(l+1)}\right]^{-1} &= (+\infty)^{-1} + \sum_{j\in\mathcal{N}(i)} \left[C_{j\to i}^{(l)}\right]^{-1} \\ &= \sum_{j\in\mathcal{N}(i)} \left[\frac{\sigma^2}{4N} + P_j^{(l)}\right]^{-1}, \end{aligned}\tag{3.24}$$

and the mean of $b_i^{(l+1)}(\theta_i)$ is obtained as

$$\begin{aligned}\mu_i^{(l+1)} &= P_i^{(l+1)}\left\{(+\infty)^{-1}\times 0 + \sum_{j\in\mathcal{N}(i)} \left[C_{j\to i}^{(l)}\right]^{-1}\nu_{j\to i}^{(l)}\right\} \\ &= P_i^{(l+1)} \sum_{j\in\mathcal{N}(i)} \left[C_{j\to i}^{(l)}\right]^{-1}\nu_{j\to i}^{(l)}, \\ &= P_i^{(l+1)} \sum_{j\in\mathcal{N}(i)} \left[\frac{\sigma^2}{4N} + P_j^{(l)}\right]^{-1}\left[\mu_j^{(l)} - \frac{1}{2N}\mathbf{1}^T\mathbf{T}_{i,j}\right], \end{aligned}\tag{3.25}$$

An equivalent form of this equation is

$$\mu_i^{(l+1)} = \frac{\sum_{j\in\mathcal{N}(i)} \left[C_{j\to i}^{(l)}\right]^{-1} \nu_{j\to i}^{(l)}}{\sum_{j\in\mathcal{N}(i)} \left[C_{j\to i}^{(l)}\right]^{-1}}. \tag{3.26}$$

It can be seen from (3.26) that each variable θ_i updates its mean of belief as weighted average of expectations from neighboring factor nodes, and expectations with small variance will contribute more to θ_i's update, and hence uncertainties due to noises and random delays are reduced.

At the end of iteration l, node i can estimate its clock offset by maximizing the belief $b_i^{(l)}(\theta_i)$ with respect to θ_i. Since $b_i^{(l)}(\theta_i)$ is Gaussian, the optimal estimation for θ_i at iteration l is given by $\hat{\theta}_i^{(I)} = \mu_i^{(l)}$. This iterative procedure is formally given in Algorithm 1.

Algorithm 1 Distributed clock synchronization using Gaussian Belief Propagation

Initialization:
Set the beliefs of the anchor nodes as $b_i(\theta_i) \sim \mathcal{N}(\theta_i; \mu_i, P_i)$, where $\mu_i = 0$ and $P_i = 0$, $i \in \mathcal{A}$
Set the beliefs of the agents nodes as $b_i^{(0)}(\theta_i) \sim \mathcal{N}(\theta_i; \mu_i^{(0)}, P_i^{(0)})$, where $\mu_i^{(0)} = 0$ and $P_i^{(0)} = +\infty$, $i \in \mathcal{M}$
Iteration until convergence:
for the l^{th} iteration **do**
  nodes $i \in \mathcal{M}$ **in parallel**
  broadcast the current belief $b_i^{(l-1)}(\theta_i)$ to neighboring nodes
  receive $b_j^{(l-1)}(\theta_j)$ from its neighboring nodes $\mathcal{N}(j)$
  update its belief $b_i^{(l)}(\theta_i) \sim \mathcal{N}(\theta_i; \mu_i^{(l)}, P_i^{(l)})$, where

$$\left[P_i^{(l)}\right]^{-1} = \sum_{j \in \mathcal{N}(i)} \left[\frac{\sigma^2}{4N} + P_j^{(l-1)}\right]^{-1}$$

  and

$$\mu_i^{(l)} = P_i^{(l)} \sum_{j \in \mathcal{N}(i)} \left[\frac{\sigma^2}{4N} + P_j^{(l-1)}\right]^{-1} \left[\mu_j^{(l-1)} - \frac{1}{2N}\mathbf{1}^T \mathbf{T}_{i,j}\right]$$

  estimate its clock offset as $\hat{\theta}_i^{(l)} = \mu_i^{(l)}$
  end parallel
end for

3.2 Non-cooperative Synchronization

In this section, we consider that there is a non zero frequency skew in the local clock readings, which needs to be estimated alongside with the time offset.

The estimation here takes place in terms of *Clients* and *Servers*: clients and servers exchange messages over a packet switched network, and we are particularly interested in the calculation of frequency offset and time offset between the clock of a client computer and the clock of a remote server. The server acts as a source of true time. It is now clear that we can face the clients as agent nodes and the servers as anchor nodes. Since the estimation is held out only with the help of the anchor nodes, we consider that we are working in a non-cooperative environment. As explained in the previous Section, we need the timestamps $\{c_i(t_1^n), c_j(t_2^n), c_j(t_3^n), c_i(t_4^n)\}_{n=1}^N$ to estimate the clock parameters of the agent nodes. Here, since always node $i \in \mathcal{M}$ and node $j \in \mathcal{A}$, we denote the timestamps as $\{C_1^n, t_2^n, t_3^n, C_4^n\}_{n=1}^N$ for simplicity.

In [4], the agent sends messages at constant intervals δT measured according to its local clock, which at true time corresponds to $\phi\delta t$, as shown in Fig. 3.2. The figure also makes it clear that the queuing delay q_1 across the forward path (from agent to anchor) is never constant and generally different from the queuing delay q_2 across the reverse path (from anchor to agent). Moreover the forward and reverse routes could be physically different; therefore, the propagations delays d_1, d_2 could be unequal across the forward and reverse paths, so

$$d_1 + q_1 \neq d_2 + q_2, \tag{3.27}$$

From Fig. 3.2, we can obtain the following relationships:

$$C_1^n - \phi t_2^n = \theta - \phi\,(d_1 + q_1)^n \tag{3.28}$$

$$C_4^n - \phi t_3^n = \theta + \phi\,(d_2 + q_2)^n \tag{3.29}$$

$$\Downarrow$$

$$C_1^n - \phi t_2^n \leq \theta - \phi d_1 \tag{3.30}$$

$$C_4^n - \phi t_3^n \geq \theta + \phi d_2. \tag{3.31}$$

To estimate the clock parameters of the agent nodes, ϕ and θ, we use Linear Programming (LP). This line-fitting technique exploits both the forward and reverse path timestamps, by estimating a clock line that minimizes the distance between the line and the data, leaving all the data points below the line on a (t_2, C_1) plane or above the line on a (t_3, C_4) plane. The following equations describe the problem and its solution:

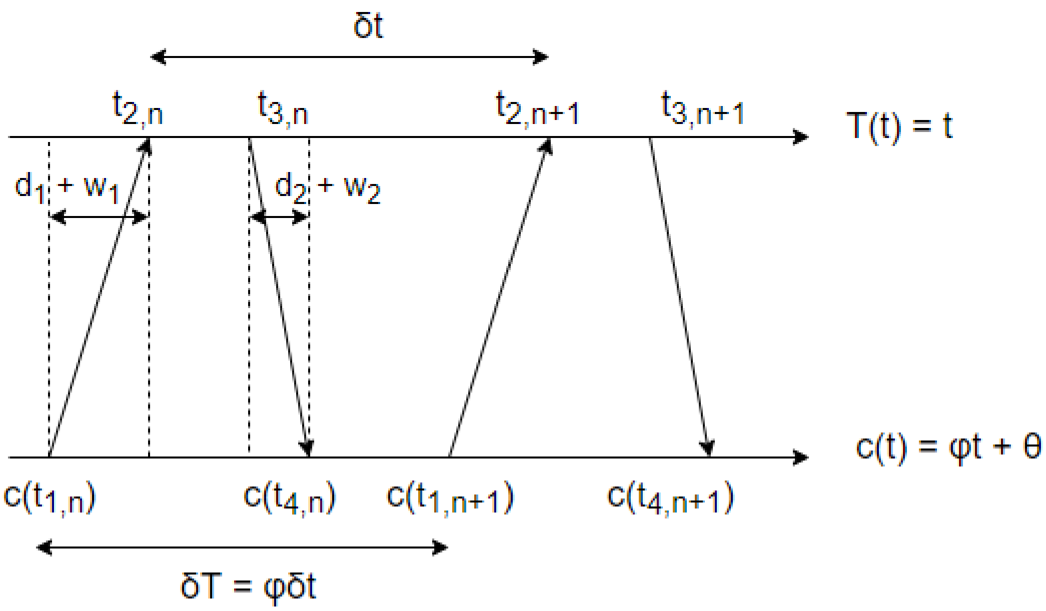

Figure 3.2: Exchanging timestamps between agent node and anchor node. Notice that a time difference δt of according to the agent's clock is translated to $\phi\delta t$ according to the anchor's clock.

1. **Forward Path:**

$$\begin{aligned} \alpha_1 &= \phi \\ \beta_1 &= \theta - \phi d_1, \end{aligned} \tag{3.32}$$

$$(3.31) \Rightarrow \alpha_1 t_2^n + \beta_1 - C_1^n \geq 0, \ \ \forall n \in [1 \dots N]. \tag{3.33}$$

Find α_1, β_1 that minimize:

$$f(\alpha_1, \beta_1) = \sum_{n=1}^{N} \left(\alpha_1 t_2^n + \beta_1 - C_1^n\right), \tag{3.34}$$

under the constraint of (3.33).

2. **Reverse Path**:

$$\begin{aligned} \alpha_2 &= \phi \\ \beta_2 &= \theta + \phi d_2, \end{aligned} \tag{3.35}$$

$$(3.32) \Rightarrow C_4^n - \alpha_2 t_3^n - \beta_2 \geq 0, \;\; \forall n \in [1 \ldots N]. \tag{3.36}$$

Find α_2, β_2 that minimize:

$$f(\alpha_2, \beta_2) = \sum_{n=1}^{N} \left(C_4^n - \alpha_2 t_3^n - \beta_2 \right), \tag{3.37}$$

under the constraint of (3.36).

Now, ϕ and θ can be estimated as:

$$\hat{\phi} = \frac{\alpha_1 + \alpha_2}{2}, \tag{3.38}$$

$$\hat{\theta} = \frac{\beta_1 + \beta_2}{2}. \tag{3.39}$$

The accuracy of the LP estimation is seriously affected by the number of packets, N, to be exchanged between the agent and anchor nodes. As exprected, the higher the number of the packets, the better the quality of the synchronization.

3.3 Cooperative Localization

At this point, we are done with the synchronization task, and we are ready to move to the localization problem, to estimate the location of the agent nodes as well. In the two following Sections, we approach this problem, as we did in the case of cooperative synchronization, by mapping the original network into a factor graph, and run two different Belief Propagation-based algorithms to obtain the posterior marginal distributions of the variable nodes, which will provide us with the location estimates of the agents in the wireless network.

3.3.1 Sum Product Algorithm in Wireless Networks - SPAWN

Here, we consider that the frequency skew is equal to 0, so we exploit the range measurements of Eq. (2.5), i.e.,

$$z_{ij,n} = ||\mathbf{x}_j - \mathbf{x}_i|| + c(\theta_j - \theta_i) + \zeta_{ij,n}, \quad n = 1, \ldots, N. \tag{3.40}$$

Assuming that the clock offset θ is estimated, we deal with the localization problem, for one round of information packet exchange for simplicity, i.e., $N = 1$. The range measurements now become

$$z_{ij} = ||\mathbf{x}_j - \mathbf{x}_i|| + \zeta_{ij}. \tag{3.41}$$

Our goal is to find the positions of all nodes in the network, $\mathbf{x}_i$, $\forall\, i \in \mathcal{M}$, based on the measurements as well as on the prior distribution of the locations of nodes, i.e. based on the the vector $\mathbf{z}$, which consists of all measurements z_{ij}, $\forall\,(i,j) \in \Xi$ and the pdf $p\,(\mathbf{x})$, respectively. The idea of the algorithm, which is developed in [1], is to factorize the posterior distribution $p\,(\mathbf{x} \mid \mathbf{z})$, in order to construct the corresponding factor graph of that factorization, and run sum-product algorithm (SPA) on it, in order to perform localization.

The application of the sum-product algorithm on that factor graph provides a distributed cooperative localization algorithm also known as sum-product algorithm over wireless networks - SPAWN.

So, at first, we simplify the factorization of global function $p\,(\mathbf{x} \mid \mathbf{z})$. More specifically, according to Bayes rule we take

$$\begin{aligned} p\,(\mathbf{x} \mid \mathbf{z}) &\propto p\,(\mathbf{x}, \mathbf{z}) \\ &= p\,(\mathbf{z} \mid \mathbf{x})\, p\,(\mathbf{x}) \\ &\overset{(*)}{=} \left(\prod_{(i,j)\in\Xi} p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j) \right) \prod_{i\in\mathcal{M}} p(\mathbf{x}_i), \end{aligned} \tag{3.42}$$

where in $(*)$ we assume that the measurements z_{ij}, $\forall(i,j) \in \Xi$ are independent, and

$$p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j) = \frac{1}{\sqrt{2\pi\sigma_d^2}} \exp\left\{-\frac{(z_{ij} - ||\mathbf{x}_j - \mathbf{x}_i||)^2}{2\sigma_d^2}\right\}. \tag{3.43}$$

Regarding the prior distributions $p(\mathbf{x}_i)$, if $i \in \mathcal{M}$ and there is not any prior knowledge available about $\mathbf{x}_i$, we consider that its prior is the uniform distribution in the network plane, i.e., $\mathbf{x}_i \sim \mathcal{U}(0, \mathbf{x}_{lim})$, where $\mathbf{x}_{lim}$ is the vector with the x-axis and y-axis limits, $[x_{lim} \;\; y_{lim}]^T$. On the other hand, if $i \in \mathcal{A}$, its prior distributions are the Dirac delta function with the real location values $\mathbf{m}_{\mathbf{x}_i} = [m_{x_i} \;\; m_{y_i}]^T$, i.e., $\delta(\mathbf{x}_i - \mathbf{m}_{\mathbf{x}_i})$.

An example of the above factorization translated into a factor graph can be seen below, in Fig. 3.3. Notice that since two agent nodes, i and j, measure the distance from one another, those measurements are completely symmetrical if they are not corrupted, e.g. by noise. So, we use just one measurement to perform localization, i.e., between $z_{i\to j}$ and $z_{j\to i}$, we only exploit $z_{i\to j}$, i.e., z_{ij} as denoted above. So, in this factor graph, it is true that $f_{ij} = f_{ji}$, which makes the sum-product algorithm and the message exchange process much simpler. This idea is also used in the next Section as well. We denote the variable of the location of each agent by X_i, i.e., $X_i = [x_i \;\; y_i]^T$, the likelihood function $p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j)$ by f_{ij} and the prior distribution $p(X_i)$ of each variable by $f_i(X_i)$.

The idea of SPAWN algorithm goes as follows: At the iteration l of the algorithm, the agent nodes broadcast their current beliefs $b_{X_i}^{(l-1)}(\cdot)$, which express the probability density function (PDF) of their location all over the network. Then all agents receive the beliefs of their neighbors and proceed to convert them to a distribution over their variable, as

$$\mu_{f_{ij}\to X_i}^{(l)}(\mathbf{x}_i) \propto \int p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j) b_{X_j}^{(l-1)}(\mathbf{x}_j) \mathrm{d}\,\mathbf{x}_j. \tag{3.44}$$

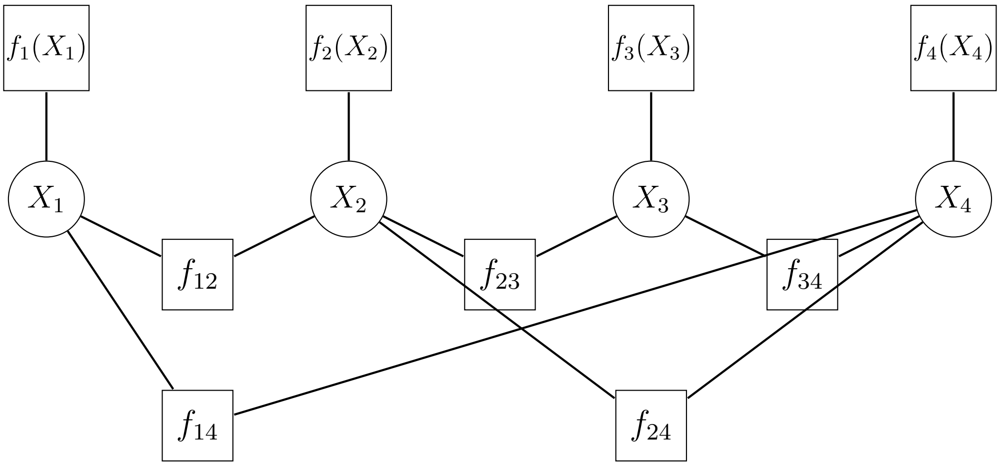

Figure 3.3: Factor graph for message exchange between 4 nodes for localization, with $f_{ij} = p(z_{ij} \mid \mathbf{x}_i, \mathbf{x}_j)$ and $f_i(X_i)$ the prior of each variable.

Finally, they update their belief as

$$b^{(l)}_{X_i}(\mathbf{x}_i) \propto \mu_{f_i \to X_i}(\mathbf{x}_i) \prod_{j \in \mathcal{N}(i)} \mu^{(l)}_{f_{ij} \to X_i}(\mathbf{x}_i), \tag{3.45}$$

where $\mu_{f_i \to X_i}$ is the message from factor node $f_i(X_i)$ to variable node X_i, representing the prior distribution of X_i, and it is constant throughout the process of the algorithm. The estimated location of every agent node i results at the last iteration N_{iter} by computing the mean squared error estimate of its belief $b^{(N_{\text{iter}})}_{X_i}(\cdot)$. Namely, the estimated location of node i is given by

$$\hat{\mathbf{X}}_i = \int_{\mathbf{x}_i} \mathbf{x}_i \, b^{(N_{iter})}_{X_i}(\mathbf{x}_i) \, \mathrm{d}\mathbf{x}_i. \tag{3.46}$$

SPAWN is illustrated in Algorithm 1.

Notice that SPAWN is a non-parametric inference algorithm, and the messages exchanged onto the corresponding factor graph are probability density functions, which means that the algorithm implementation must be sample-based. This requires a lot of samples in order for the estimation to be efficient and results to huge communication overhead. So, a need for a more efficient and less expensive algorithm is arising.

Algorithm 2 Cooperative Localization with SPAWN

Initialization:
nodes $i \in \mathcal{A}$ **in parallel** *initialize*
    belief $b^{(0)}_{X_i}(\cdot) = \mu_{f_i \to X_i} = \delta(\mathbf{x}_i - \mathbf{m}_{\mathbf{x}i})$
nodes $i \in \mathcal{M}$ **in parallel** *initialize*
    belief $b^{(0)}_{X_i}(\cdot) = \mu_{f_i \to X_i} \sim \mathcal{U}(0, \mathbf{x}_{lim})$
end parallel
for $l = 1$ to N_{iter} **do**
    nodes $i \in \mathcal{M}$ **in parallel**
        broadcast $b^{(l-1)}_{X_i}(\cdot)$
        receive $b^{(l-1)}_{X_j}(\cdot)$ from neighbors $j \in \mathcal{N}(i)$
        convert $b^{(l-1)}_{X_j}(\cdot)$to a distribution over $\mathbf{x}_i$ as

$$\mu^{(l)}_{f_{ij} \to X_i}(\mathbf{x}_i) \propto \int p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j) b^{(l-1)}_{X_j}(\mathbf{x}_j) \mathrm{d}\, \mathbf{x}_j$$

        compute new messages as

$$b^{(l)}_{X_i}(\mathbf{x}_i) \propto \mu_{f_i \to X_i}(\mathbf{x}_i) \prod_{j \in \mathcal{N}(i)} \mu^{(l)}_{f_{ij} \to X_i}(\mathbf{x}_i)$$

    end parallel
    estimate the agents' position using the **MMSE** estimator
end for

3.3.2 Belief Propagation with Taylor expansion

In a similar manner with the previous Section about SPAWN, we continue our analysis considering that the frequency skew is equal to 0 and exploiting the range measurements of Eq. (2.5). We still assume that the clock offset θ is estimated, and that we are working for one round of information packet exchange, i.e., $N = 1$ so the range measurements are still in the form of

$$z_{ij} = ||\mathbf{x}_j - \mathbf{x}_i|| + \zeta_{ij}. \tag{3.47}$$

Recall that if $\mathbf{X} = \{\mathbf{x}_i \,|\, i \in \mathcal{M}\}$ and $\mathbf{Z} = \{z_{ij} \,|\, (i,j) \in \Xi\}$, then the joint posterior distribution can be calculated by the Bayesian Theorem as

$$\begin{aligned} p(\mathbf{X}|\mathbf{Z}) &\propto p(\mathbf{Z}|\mathbf{X})\, p(\mathbf{X}) \\ &\overset{(*)}{\propto} \left(\prod_{(i,j)\in\Xi} p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j) \right) \prod_{i\in\mathcal{M}} p(\mathbf{x}_i), \end{aligned} \tag{3.48}$$

where the likelihood function $p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j)$ is given by

$$\begin{aligned} p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j) &= \frac{1}{\sqrt{2\pi\sigma_d^2}} \exp\left\{ -\frac{(z_{ij} - ||\mathbf{x}_j - \mathbf{x}_i||)^2}{2\sigma_d^2} \right\} \\ &\propto \exp\left\{ -\frac{(z_{ij} - ||\mathbf{x}_j - \mathbf{x}_i||)^2}{2\sigma_d^2} \right\} \\ &\propto \exp\left\{ -\frac{z_{ij}^2 - 2z_{ij}\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} + (x_j - x_i)^2 + (y_j - y_i)^2}{2\sigma_d^2} \right\}, \end{aligned} \tag{3.49}$$

where in $(*)$ we still assume that the measurements z_{ij}, $\forall (i,j) \in \Xi$ are independent. Here, the priors $p(\mathbf{x}_i)$ are assumed Gaussian if $i \in \mathcal{M}$, while if $i \in \mathcal{A}$, meaning for the anchor node i there are no location uncertainties, the prior distributions are the Dirac delta function, which can also be expressed as Gaussian distribution with zero variance.

To obtain the estimate of the location of the agent node i, we need the marginal posterior distribution

$$p(\xi_i|\mathbf{Z}) \propto \int p(\mathbf{X}|\mathbf{Z}) \sim \{d\xi_i\}, \tag{3.50}$$

with $\xi_i \in \{x_i, y_i\}$ and $\sim \{d\xi_i\}$ denoting the integration over all variables collected in $\mathbf{X}$, except for the variable ξ_i. Then, we use the MMSE estimator to get the estimate

$$\hat{\xi}_i = \int \xi_i \, p(\xi_i|\mathbf{Z}) \, d\xi_i, \tag{3.51}$$

Direct marginalization of (3.48) as (3.50) is hard to calculate, so an efficient way to solve this problem is to exploit the factor graph and the message passing approach.

Assuming that the x-axis and y-axis coordinates are independent, we can rewrite the posterior distibution as

$$p(\mathbf{x}_i|z_{ij}, \forall j \in \mathcal{N}(i)) \propto p(x_i)p(y_i) \prod_{j\in\mathcal{N}(i)} p(z_{ij} \,|\, x_i, x_j, y_i, y_j), \tag{3.52}$$

The joint distribution is factorized and, thus, can be represented by a factor graph as depicted in Fig. 3.4, which shows a single adjacency between two nodes, i and j, for brevity.

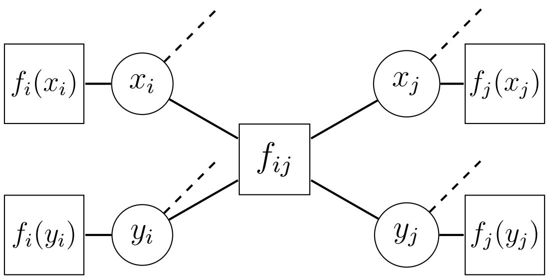

Figure 3.4: Factor graph for message exchange between Node i and Node j for localization, with $f_{ij} = p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j)$ and $f(\xi)$ the prior of each variable. The dashed lines denote that the variable nodes have more neighbors.

We are now able to run Belief Propagation on the aforementioned factor graph to obtain the beliefs $b(\xi_i)$, which approximate the marginals $p(\xi_i \,|\, \mathbf{Z})$, $\xi_i \in \{x_i, y_i\}$, $i \in \mathcal{M}$. There are two kinds of messages being exchanged in the factor graph, messages from a factor node to a variable node and messages from variable node to factor node. At the l-th iteration, the messages from factor to variable node are given by

$$\begin{aligned} \mu^{(l)}_{f_{ij}\to x_i}(x_i) = \int\!\!\int\!\!\int & f_{ij}(x_i, y_i, x_j, y_j)\, \mu^{(l-1)}_{x_j\to f_{ij}}(x_j)\, \mu^{(l-1)}_{y_i\to f_{ij}}(y_i) \\ & \times \mu^{(l-1)}_{y_j\to f_{ij}}(y_j)\, dx_j\, dy_i\, dy_j, \end{aligned} \tag{3.53}$$

$$\mu^{(l)}_{f_{ij}\to y_i}(y_i) = \int\int\int f_{ij}(x_i, y_i, x_j, y_j)\, \mu^{(l-1)}_{y_j\to f_{ij}}(y_j)\, \mu^{(l-1)}_{x_i\to f_{ij}}(x_i) \times \mu^{(l-1)}_{x_j\to f_{ij}}(x_j)\, dy_j\, dx_i\, dx_j, \tag{3.54}$$

$$\mu_{f_i\to\xi_i}(\xi_i) = p(\xi_i), \tag{3.55}$$

with $\xi_i \in \{x_i, y_i\}$, $f_{ij}(x_i, y_i, x_j, y_j) \triangleq p(z_{ij}\,|\,\mathbf{x}_i, \mathbf{x}_j)$, and $p(\xi_i)$ the prior distribution of the variable ξ_i. Notice that the message $\mu_{f_i\to\xi_i}(\xi_i)$ is constant throughout the algorithm execution.

The belief of each variable ξ_i is given by

$$b^{(l)}(\xi_i) = \mu_{f_i\to\xi_i}(\xi_i) \prod_{j\in\mathcal{N}(i)} \mu^{(l-1)}_{f_{ij}\to\xi_i}(\xi_i), \tag{3.56}$$

and, finally, the message from variable to factor node can be calculated as $\frac{b^{(l)}(\xi_i)}{\mu^{(l)}_{f_{ij}\to\xi_i}(\xi_i)}$, i.e.,

$$\mu^{(l)}_{\xi_i\to f_{ij}}(\xi_i) = \mu_{f_i\to\xi_i}(\xi_i) \prod_{j'\in\mathcal{N}(i)/j} \mu^{(l-1)}_{f_{ij'}\to\xi_i}(\xi_i), \tag{3.57}$$

A problem that arises at this point is that the computation of the messages (3.53) and (3.54) is intractable due to the nonlinear terms in the likelihood function f_{ij}. In this paper, we approach this problem by using the first order of Taylor expansion [3] to linearize the norm in the f_{ij} function and, thus, manage to update all messages in closed-form Gaussian expressions.

At the l-th iteration, we expand the square root term in Eq. (3.49), according to Taylor series around node i's and node j's location estimations $(\hat{x}_i^{(l-1)}, \hat{y}_i^{(l-1)})$ and $(\hat{x}_j^{(l-1)}, \hat{y}_j^{(l-1)})$ of the previous iteration:

$$\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \simeq \hat{d}_{ij}^{(l-1)} + \lambda_{ij}^{(l-1)}(x_i - \hat{x}_i^{(l-1)}) + \gamma_{ij}^{(l-1)}(y_i - \hat{y}_i^{(l-1)}) + \lambda_{ij}^{(l-1)}(\hat{x}_j^{(l-1)} - x_j) + \gamma_{ij}^{(l-1)}(\hat{y}_j^{(l-1)} - y_j), \tag{3.58}$$

with $\hat{d}_{ij}^{(l-1)} = \sqrt{(\hat{x}_j^{(l-1)} - \hat{x}_i^{(l-1)})^2 + (\hat{y}_j^{(l-1)} - \hat{y}_i^{(l-1)})^2}$ the range estimate, and

$\lambda_{ij}^{(l-1)} = \frac{\hat{x}_i^{(l-1)} - \hat{x}_j^{(l-1)}}{\hat{d}_{ij}^{(l-1)}}$, $\gamma_{ij}^{(l-1)} = \frac{\hat{y}_i^{(l-1)} - \hat{y}_j^{(l-1)}}{\hat{d}_{ij}^{(l-1)}}$ the directional derivatives on x-axis and y-axis, respectively. Now this approximation can be simplified as:

$$
\begin{aligned}
(3.58) &= \hat{d}_{ij}^{(l-1)} + \lambda_{ij}^{(l-1)}(x_i - x_j) + \gamma_{ij}^{(l-1)}(y_i - y_j) \\
&\quad - \lambda_{ij}^{(l-1)}(\hat{x}_i^{(l-1)} - \hat{x}_j^{(l-1)}) - \gamma_{ij}^{(l-1)}(\hat{y}_i^{(l-1)} - \hat{y}_j^{(l-1)}) \\
&= \hat{d}_{ij}^{(l-1)} + \lambda_{ij}^{(l-1)}(x_i - x_j) + \gamma_{ij}^{(l-1)}(y_i - y_j) \\
&\quad - \left(\frac{(\hat{x}_i^{(l-1)} - \hat{x}_j^{(l-1)})^2}{\hat{d}_{ij}^{(l-1)}} + \frac{(\hat{y}_i^{(l-1)} - \hat{y}_j^{(l-1)})^2}{\hat{d}_{ij}^{(l-1)}} \right) \\
&= \hat{d}_{ij}^{(l-1)} + \lambda_{ij}^{(l-1)}(x_i - x_j) + \gamma_{ij}^{(l-1)}(y_i - y_j) - \frac{(\hat{d}_{ij}^{(l-1)})^2}{\hat{d}_{ij}^{(l-1)}} \\
&= \lambda_{ij}^{(l-1)}(x_i - x_j) + \gamma_{ij}^{(l-1)}(y_i - y_j). \qquad (3.59)
\end{aligned}
$$

We are now able to use (3.59) in (3.49) and evaluate the messages in (3.53) and (3.54), assuming that the messages $\mu_{\xi_i \to f_{ij}}^{(l-1)}(\xi_i)$ are Gaussian, i.e.,

$$
\mu_{\xi_i \to f_{ij}}^{(l-1)}(\xi_i) \propto \mathcal{N}\left(\xi_i, m_{\xi_i \to f_{ij}}^{(l-1)}, \left(\sigma_{\xi_i \to f_{ij}}^{(l-1)}\right)^2\right), \forall i \in \mathcal{M} \cup \mathcal{A}. \qquad (3.60)
$$

Notice that the expressions for $\mu_{f_{ij} \to x_i}^{(l)}$ and $\mu_{f_{ij} \to y_i}^{(l)}$ are symmetrical, so we only present the derivation for $\mu_{f_{ij} \to x_i}^{(l)}$ in Section 7.1 of the Appendix Chapter. 7.2. The final expressions of the messages in Gaussian form, i.e., $\mu_{f_{ij} \to \xi_i}^{(l)}(\xi_i) \propto \mathcal{N}\left(\xi_i, m_{f_{ij} \to \xi_i}^{(l)}, \left(\sigma_{f_{ij} \to \xi_i}^{(l)}\right)^2\right)$, are given by:

$$
m_{f_{ij} \to x_i}^{(l)} = m_{x_j \to f_{ij}}^{(l-1)} + z_{ij}\lambda_{ij}^{(l-1)}, \qquad (3.61)
$$

$$
\left(\sigma_{f_{ij} \to x_i}^{(l)}\right)^2 = \left(\sigma_{x_j \to f_{ij}}^{(l-1)}\right)^2 + \sigma_d^2, \qquad (3.62)
$$

$$
m_{f_{ij} \to y_i}^{(l)} = m_{y_j \to f_{ij}}^{(l-1)} + z_{ij}\gamma_{ij}^{(l-1)}, \qquad (3.63)
$$

$$
\left(\sigma_{f_{ij} \to y_i}^{(l)}\right)^2 = \left(\sigma_{y_j \to f_{ij}}^{(l-1)}\right)^2 + \sigma_d^2. \qquad (3.64)
$$

When all the incoming messages from connected factor vertices are collected, we can calculate the belief of the variable ξ_i with Eq. 3.57. Since the incoming messages are all Gaussian, the product of multiple Gaussians is still a Gaussian, according to the following theorem.

***Theorem* 1.** *The product of multiple Gaussian PDFs of the same variable x, i.e.,*

$$\prod_{i=1}^{N} \mathcal{N}\left(x;\, m_i, \sigma_i^2\right), \tag{3.65}$$

is a Gaussian PDF, i.e., $\mathcal{N}\left(x;\, m', (\sigma')^2\right)$ *with variance*

$$(\sigma')^2 = \left(\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right)^{-1}, \tag{3.66}$$

and mean value

$$m' = (\sigma')^2 \sum_{i=1}^{N} \frac{m_i}{\sigma_i^2}. \tag{3.67}$$

According to Theorem 1, the beliefs of the variables $\xi_i \in \{x_i, y_i\}$ are Gaussian, i.e.,

$$b^{(l)}(\xi_i) \propto \mathcal{N}\left(\xi_i,\, m_{\xi_i}^{(l)}, \left(\sigma_{\xi_i}^{(l)}\right)^2\right), \tag{3.68}$$

with

$$m_{\xi_i}^{(l)} = \left(\sigma_{\xi_i}^{(l)}\right)^2 \left(\frac{m_{f_i \to \xi_i}}{\sigma_{f_i \to \xi_i}^2} + \sum_{j \in \mathcal{N}(i)} \frac{m_{f_{ij} \to \xi_i}^{(l)}}{\left(\sigma_{f_{ij} \to \xi_i}^{(l)}\right)^2}\right), \tag{3.69}$$

$$\left(\sigma_{\xi_i}^{(l)}\right)^2 = \left(\frac{1}{\sigma_{f_i \to \xi_i}^2} + \sum_{j \in \mathcal{N}(i)} \frac{1}{\left(\sigma_{f_{ij} \to \xi_i}^{(l)}\right)^2}\right)^{-1}. \tag{3.70}$$

Notice that since the beliefs are Gaussian, the MMSE estimator of each variable is the mean value of its belief.

Finally, the messages of Eq. 3.56 can be calculated as

$$\mu^{(l)}_{\xi_i \to f_{ij}}(\xi_i) = \frac{b^{(l)}(\xi_i)}{\mu^{(l)}_{f_{ij} \to \xi_i}(\xi_i)}, \tag{3.71}$$

which is also Gaussian according to Theorem 2, i.e.,

$$\mu^{(l)}_{\xi_i \to f_{ij}}(\xi_i) \propto \mathcal{N}\Big(\xi_i,\, m^{(l)}_{\xi_i \to f_{ij}},\, \Big(\sigma^{(l)}_{\xi_i \to f_{ij}}\Big)^2\Big), \tag{3.72}$$

with

$$m^{(l)}_{\xi_i \to f_{ij}} = \frac{m^{(l)}_{\xi_i}\Big(\sigma^{(l)}_{f_{ij} \to \xi_i}\Big)^2 - m^{(l)}_{f_{ij} \to \xi_i}\Big(\sigma^{(l)}_{\xi_i}\Big)^2}{\Big(\sigma^{(l)}_{f_{ij} \to \xi_i}\Big)^2 - \Big(\sigma^{(l)}_{\xi_i}\Big)^2}, \tag{3.73}$$

$$\Big(\sigma^{(l)}_{\xi_i \to f_{ij}}\Big)^2 = \frac{\Big(\sigma^{(l)}_{f_{ij} \to \xi_i}\Big)^2\Big(\sigma^{(l)}_{\xi_i}\Big)^2}{\Big(\sigma^{(l)}_{f_{ij} \to \xi_i}\Big)^2 - \Big(\sigma^{(l)}_{\xi_i}\Big)^2}. \tag{3.74}$$

***Theorem* 2.** *The quotient of two Gaussian PDFs of the same variable x, i.e.,*

$$\frac{\mathcal{N}\left(x;\, m_1, \sigma_1^2\right)}{\mathcal{N}\left(x;\, m_2, \sigma_2^2\right)}, \tag{3.75}$$

is a Gaussian PDF, i.e., $\mathcal{N}\left(x;\, m', (\sigma')^2\right)$ with mean value

$$m' = \frac{m_1\sigma_2^2 - m_2\sigma_1^2}{\sigma_2^2 - \sigma_1^2}, \tag{3.76}$$

and variance

$$(\sigma')^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_2^2 - \sigma_1^2}. \tag{3.77}$$

The Localization algorithm is summed up in Algorithm 3.

Algorithm 3 Localization with Belief Propagation

Initialization:
nodes $i \in \mathcal{A}$ **in parallel** *initialize*
    $f_i(x_i) = \delta(x_i - m_{x_i}),\ f_i(y_i) = \delta(y_i - m_{y_i})$
nodes $i \in \mathcal{M}$ **in parallel** *initialize*
    $f_i(x_i) \sim \mathcal{N}(x_i; 0, \sigma^2_{x_i}),\ f_i(y_i) \sim \mathcal{N}(y_i; 0, \sigma^2_{y_i})$
end parallel
for $l = 1$ to N_{iter} **do**
    nodes $i \in \mathcal{M}$ **in parallel**
        compute all messages from factor nodes to variable nodes according to (3.61) - (3.64)
        update means and variances of beliefs according to (3.69) and (3.70) respectively
        compute all messages from variable nodes to neighboring nodes (3.73) and (3.74)
    end parallel
    estimate the agents' position using the **MMSE** estimator
end for

In comparison with the prior art, the solution of the localization problem with the approximate GBP has also been studied in [5]. The main difference with the aforementioned algorithm is the factor graph this work uses to run the GBP. From Fig. 3.5, it can be seen that the corresponding FG between two nodes i and m includes both factors $h_{m \to i}$ and $h_{i \to m}$. In our case, that would mean that apart from the factor f_{ij}, we would also need the factor f_{ji}, which in practice means that we do not only need the range measurement z_{ij}, but the measurement z_{ji} as well. This round-trip information, though, could be unavailable in many wireless networks. Due the symmetry of the range measurements, we are able to exploit just one of them to create the factor graph and run GBP on it, so we manage to make the FG and, thus, the message passing scheme simpler. Notice that the measurements z_{ij}, z_{ji} are not completely symmetrical because the distance measurement is affected by the system noise. Nevertheless, the idea to use the one-way measurements for the estimation remains valid and applicable.

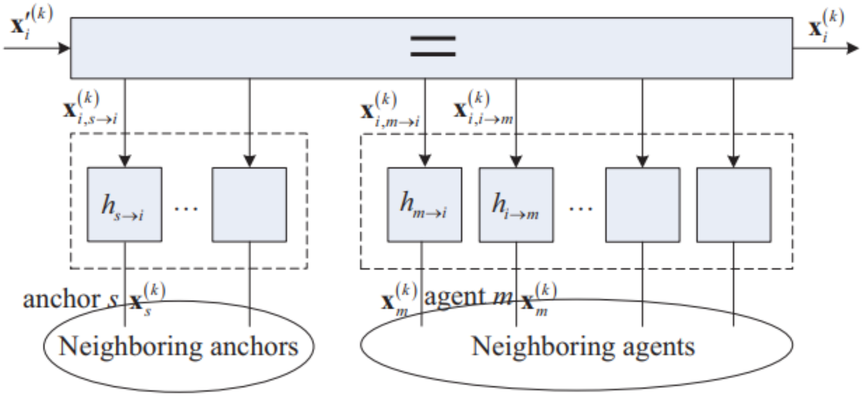

Figure 3.5: Factor graph of message passing between node i and its neighbors in [5].

Chapter 4

Joint Synchronization and Localization

Our aim in this Chapter is to perform spatio-temporal estimation, not in a sequential manner, but try to estimate the time offset and location of the agent nodes simultaneously. We keep on following the same process by running approximate Gaussian Belief Propagation on the corresponding factor graph.

4.1 Joint Non-cooperative Estimation

Here, we use the measurement model of Eq.(2.5), i.e.,

$$z_{ij,n} = ||\mathbf{x}_j - \mathbf{x}_i|| + c(\theta_j - \theta_i) + \zeta_{ij,n}, \quad n = 1, \ldots, N, \tag{4.1}$$

to estimate the clock offsets and the locations of the agent nodes with Belief Propagation, jointly. Again, we present the analysis considering one information packet for simplicity and clarity, so $N = 1$. We consider a non-cooperative environment, which means that the agents exchange messages only with anchor nodes. So, wherever we refer to node j in this Section, we mean the **anchor** node j.

The likelihood function $f_{ij}(x_i, y_i, \theta_i, x_j, y_j, \theta_j) \triangleq p(z_{ij} \,|\, \mathbf{x}_i, \mathbf{x}_j, \theta_i, \theta_j)$ for the joint case is

$$f_{ij} \propto \exp\left\{-\frac{(z_{ij} - ||\mathbf{x}_j - \mathbf{x}_i|| - c(\theta_j - \theta_i))^2}{2\sigma_d^2}\right\}, \tag{4.2}$$

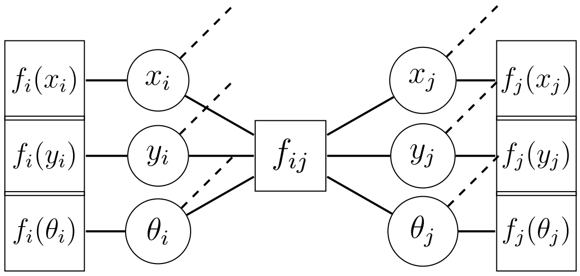

Figure 4.1: Factor graph for message exchange between agent Node i and anchor Node j for joint localization and synchronization.

and the joint posterior distribution is

$$p(\mathbf{x}_i, \theta_i | z_{ij}, \forall j \in \mathcal{N}(i)) \propto \\ p(x_i)p(y_i)p(\theta_i) \prod_{j \in \mathcal{N}(i)} p(z_{ij} \,|\, x_i, x_j, y_i, y_j, \theta_i, \theta_j). \tag{4.3}$$

So, the factor graph of the message passing now becomes as the one in Fig. (4.1), and the corresponding messages from the factor node f_{ij} to the variable node ξ_i with $\xi_i \in \{x_i, y_i, \theta_i\}$ are calculated as:

$$\mu^{(l)}_{f_{ij} \to \xi_i}(\xi_i) = \int \cdots \int f_{ij}(x_i, y_i, \theta_i, x_j, y_j, \theta_j) \prod_{\vartheta \in \mathcal{F}_{i,j}/\xi_i} \mu^{(l-1)}_{\vartheta \to f_{ij}}(\vartheta)\, d\vartheta, \tag{4.4}$$

where $\mathcal{F}_{i,j}$ denotes the set of the variables connected to the factor f_{ij}, i.e., $\mathcal{F}_{i,j} = \{x_i, y_i, \theta_i, x_j, y_j, \theta_j\}$. Since the agents receive messages only from anchors, the messages $\mu^{(l-1)}_{\xi_j \to f_{ij}}$ are equal to the prior distribution of each variable of each anchor, $\delta(\xi_j - m_{\xi_j})$, with $\xi_j \in \{x_j, y_j, \theta_j\}$ and m_{ξ_j} the real value of the variable ξ_j.

We now follow the same process as in the cooperative environment to get the mathematical expressions of the $\mu^{(l)}_{f_{ij} \to \xi_i}$ messages, with $\xi_i \in \{x_i, y_i, \theta_i\}$. We provide the equations of the messages in the Gaussian information form, i.e.,

$$\mathcal{N}^{-1}\left(\xi_i; \mathrm{h}^{(l)}_{f_{ij}\to\xi_i}, \mathrm{J}^{(l)}_{f_{ij}\to\xi_i}\right) \propto \exp\left\{-\frac{1}{2}\,\mathrm{J}^{(l)}_{f_{ij}\to\xi_i}\,\xi_i^2 + \mathrm{h}^{(l)}_{f_{ij}\to\xi_i}\,\xi_i\right\}, \tag{4.5}$$

so the mean values and the variances can be calculated as:

$$m^{(l)}_{f_{ij}\to\xi_i} = \frac{\mathrm{h}^{(l)}_{f_{ij}\to\xi_i}}{\mathrm{J}^{(l)}_{f_{ij}\to\xi_i}}, \tag{4.6}$$

$$\left(\sigma^{(l)}_{f_{ij}\to\xi_i}\right)^2 = \frac{1}{\mathrm{J}^{(l)}_{f_{ij}\to\xi_i}}. \tag{4.7}$$

The final equations of the messages $\mu^{(l)}_{f_{ij}\to x_i}$, $\mu^{(l)}_{f_{ij}\to y_i}$, $\mu^{(l)}_{f_{ij}\to\theta_i}$ are given in Eqs. (4.8), (4.9), (4.10), respectively, as follows:

$$\mu^{(l)}_{f_{ij}\to x_i}(x_i) \propto$$

$$\exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_d^2} + \left(\lambda_{ij}^{(l-1)}\right)^2\left(\frac{1}{\sigma_d^2 + c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2} - \frac{1}{\sigma_d^2}\right)\right.\right.$$

$$\left. - \left(\lambda_{ij}^{(l-1)}\right)^2\left(\gamma_{ij}^{(l-1)}\right)^2\left(\frac{1}{\sigma_d^2 + c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2} - \frac{1}{\sigma_d^2}\right)^2\left(\frac{\sigma_d^2\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}{\sigma_d^2 + \left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}\right)\right] x_i^2$$

$$+ \left[\frac{1}{\sigma_d^2}\left(m_{x_j} + \lambda_{ij}^{(l-1)} z_{ij}\right) + \lambda_{ij}^{(l-1)}\left(\frac{1}{\sigma_d^2 + c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2} - \frac{1}{\sigma_d^2}\right)\times\right.$$

$$\left(z_{ij} + m_{x_j}\lambda_{ij}^{(l-1)} + m_{y_j}\gamma_{ij}^{(l-1)}\right) + \frac{c\lambda_{ij}^{(l-1)} m^{(l-1)}_{\theta_i\to f_{ij}}}{\sigma_d^2 + c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2}$$

$$-\lambda_{ij}^{(l-1)}\gamma_{ij}^{(l-1)}\left(\frac{1}{\sigma_d^2 + c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2} - \frac{1}{\sigma_d^2}\right)\left(\frac{\sigma_d^2\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}{\sigma_d^2 + \left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}\right)\times$$

$$\left(\frac{m^{(l-1)}_{y_i\to f_{ij}}}{\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}+\frac{1}{\sigma_d^2}\left(m_{y_j}+z_{ij}\gamma^{(l-1)}_{ij}\right)\right)\Bigg]\, x_i\Bigg\}, \tag{4.8}$$

$$\begin{aligned}
&\mu^{(l)}_{f_{ij}\to y_i}(y_i)\propto\\
&\exp\Bigg\{-\frac{1}{2}\Bigg[\frac{1}{\sigma_d^2}+\left(\gamma^{(l-1)}_{ij}\right)^2\left(\frac{1}{\sigma_d^2+c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2}-\frac{1}{\sigma_d^2}\right)\\
&\quad-\left(\gamma^{(l-1)}_{ij}\right)^2\left(\lambda^{(l-1)}_{ij}\right)^2\left(\frac{1}{\sigma_d^2+c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2}-\frac{1}{\sigma_d^2}\right)^2\left(\frac{\sigma_d^2\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2}{\sigma_d^2+\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2}\right)\Bigg]\, y_i^2\\
&\quad+\Bigg[\frac{1}{\sigma_d^2}\left(m_{y_j}+\gamma^{(l-1)}_{ij}z_{ij}\right)+\gamma^{(l-1)}_{ij}\left(\frac{1}{\sigma_d^2+c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2}-\frac{1}{\sigma_d^2}\right)\times\\
&\qquad\left(z_{ij}+m_{y_j}\gamma^{(l-1)}_{ij}+m_{x_j}\lambda^{(l-1)}_{ij}\right)+\frac{c\gamma^{(l-1)}_{ij}m^{(l-1)}_{\theta_i\to f_{ij}}}{\sigma_d^2+c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2}\\
&\qquad-\gamma^{(l-1)}_{ij}\lambda^{(l-1)}_{ij}\left(\frac{1}{\sigma_d^2+c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2}-\frac{1}{\sigma_d^2}\right)\left(\frac{\sigma_d^2\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2}{\sigma_d^2+\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2}\right)\times\\
&\qquad\left(\frac{m^{(l-1)}_{x_i\to f_{ij}}}{\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2}+\frac{1}{\sigma_d^2}\left(m_{x_j}+z_{ij}\lambda^{(l-1)}_{ij}\right)\right)\Bigg]\, y_i\Bigg\},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\mu^{(l)}_{f_{ij}\to\theta_i}(\theta_i)\propto\\
&\exp\Bigg\{-\frac{1}{2}\left[\frac{c^2\left(\sigma_d^2+\left(\gamma^{(l-1)}_{ij}\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2+\left(\lambda^{(l-1)}_{ij}\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2\right)}{\left(\sigma_d^2+\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2\right)\left(\sigma_d^2+\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2\right)}\right]\theta_i^2
\end{aligned}$$

$$+\left[\frac{c\lambda_{ij}^{(l-1)}m_{x_i\to f_{ij}}^{(l-1)}}{\sigma_d^2+\left(\sigma_{x_i\to f_{ij}}^{(l-1)}\right)^2}+\frac{c\gamma_{ij}^{(l-1)}m_{y_i\to f_{ij}}^{(l-1)}}{\sigma_d^2+\left(\sigma_{y_i\to f_{ij}}^{(l-1)}\right)^2}+\frac{c\lambda_{ij}^{(l-1)}\left(\sigma_{x_i\to f_{ij}}^{(l-1)}\right)^2}{\sigma_d^2\left(\sigma_d^2+\left(\sigma_{x_i\to f_{ij}}^{(l-1)}\right)^2\right)}\times\right.$$
$$\left(m_{x_j}+\lambda_{ij}^{(l-1)}z_{ij}\right)+\frac{c\gamma_{ij}^{(l-1)}\left(\sigma_{y_i\to f_{ij}}^{(l-1)}\right)^2}{\sigma_d^2\left(\sigma_d^2+\left(\sigma_{y_i\to f_{ij}}^{(l-1)}\right)^2\right)}\left(m_{y_j}+\gamma_{ij}^{(l-1)}z_{ij}\right)$$
$$\left.-\frac{c}{\sigma_d^2}\left(z_{ij}+m_{x_j}\lambda_{ij}^{(l-1)}+m_{y_j}\gamma_{ij}^{(l-1)}\right)\right]\theta_i\Bigg\}. \tag{4.10}$$

The beliefs and the messages from variable to factor nodes can be calculated exactly as described in the previous Chapter, using the Eqs. 3.69, 3.70, 3.73 and 3.74.

4.2 Joint Cooperative Estimation

After solving the joint estimation problem in a non-cooperative way, it is expected to proceed to the derivation of a cooperative algorithm, where we consider that agents exchange messages with each other as well, not only with anchors.

Again, taking advantage of the Taylor expansion, we present the equations of the messages $\mu_{f_{ij}\to\xi_i}^{(l)}$ in Gaussian form, as they were presented in the [3], i.e.,

$$\mu_{f_{ij}\to\xi_i}^{(l)}(\xi_i)\propto\mathcal{N}\left(\xi_i,m_{f_{ij}\to\xi_i}^{(l)},\left(\sigma_{f_{ij}\to\xi_i}^{(l)}\right)^2\right), \tag{4.11}$$

If node j is an anchor node, we have the parameters for the location coordinates as

$$m^{(l)}_{f_{ij}\to x_i} = m_{x_j} + \left(z_{ij} - cm^{(l-1)}_{\theta_i\to f_{ij}}\right)\lambda^{(l-1)}_{ij}, \tag{4.12}$$

$$m^{(l)}_{f_{ij}\to y_i} = m_{y_j} + \left(z_{ij} - cm^{(l-1)}_{\theta_i\to f_{ij}}\right)\gamma^{(l-1)}_{ij}, \tag{4.13}$$

$$\left(\sigma^{(l)}_{f_{ij}\to x_i}\right)^2 = \left(\sigma^{(l)}_{f_{ij}\to y_i}\right)^2 = c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2 + \sigma_d^2, \tag{4.14}$$

and the parameters for the clock offset as

$$\begin{aligned} m^{(l)}_{f_{ij}\to\theta_i} = \frac{z_{ij}}{c} - \frac{1}{c^2\left(\sigma^{(l-1)}_{xyd}\right)^2}\Bigg[&\left(\sigma^{(l-1)}_{xd}\right)^2\left(\sigma^{(l-1)}_{yd}\right)^2 \hat{d}^{(l-1)}_{ij} \\ &- \left(\sigma^{(l-1)}_{xd}\right)^2\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2\gamma^{(l-1)}_{ij}\left(m^{(l-1)}_{x_i\to f_{ij}} - m_{x_j}\right) \\ &+ \left(\sigma^{(l-1)}_{yd}\right)^2\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2\lambda^{(l-1)}_{ij}\left(m^{(l-1)}_{y_i\to f_{ij}} - m_{y_j}\right)\Bigg], \end{aligned} \tag{4.15}$$

$$\left(\sigma^{(l)}_{f_{ij}\to\theta_i}\right)^2 = \frac{\left(\sigma^{(l-1)}_{xd}\right)^2\left(\sigma^{(l-1)}_{yd}\right)^2}{c^2\left(\sigma^{(l-1)}_{xyd}\right)^2}, \tag{4.16}$$

with $\left(\sigma^{(l-1)}_{xd}\right)^2 \triangleq \left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2 + \sigma^2_{d'}$, $\left(\sigma^{(l-1)}_{yd}\right)^2 \triangleq \left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2 + \sigma^2_d$ and $\left(\sigma^{(l-1)}_{xyd}\right)^2 \triangleq \sigma^2_d + \left(\sigma^{(l-1)}_{x_i\to f_{ij}}\lambda^{(l-1)}_{ij}\right)^2 + \left(\sigma^{(l-1)}_{y_i\to f_{ij}}\gamma^{(l-1)}_{ij}\right)^2$.

If node j is an agent node, the parameters for location coordinates are determined as

$$m^{(l)}_{f_{ij}\to x_i} = m^{(l-1)}_{x_j\to f_{ij}} + \left(z_{ij} - c\left(m^{(l-1)}_{\theta_i\to f_{ij}} - m^{(l-1)}_{\theta_j\to f_{ij}}\right)\right)\lambda^{(l-1)}_{ij}, \tag{4.17}$$

$$m^{(l)}_{f_{ij}\to y_i} = m^{(l-1)}_{y_j\to f_{ij}} + \left(z_{ij} - c\left(m^{(l-1)}_{\theta_i\to f_{ij}} - m^{(l-1)}_{\theta_j\to f_{ij}}\right)\right)\gamma^{(l-1)}_{ij}, \tag{4.18}$$

$$\left(\sigma^{(l)}_{f_{ij}\to x_i}\right)^2 = c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2 + c^2\left(\sigma^{(l-1)}_{\theta_j\to f_{ij}}\right)^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2 + \sigma^2_d, \tag{4.19}$$

$$\left(\sigma^{(l)}_{f_{ij}\to y_i}\right)^2 = c^2\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2 + c^2\left(\sigma^{(l-1)}_{\theta_j\to f_{ij}}\right)^2 + \left(\sigma^{(l-1)}_{y_j\to f_{ij}}\right)^2 + \sigma^2_d, \tag{4.20}$$

and the parameters for the clock offset as

$$\begin{aligned} m^{(l)}_{f_{ij}\to\theta_i} =& \frac{z_{ij}}{c} - \frac{1}{c^2\left(\sigma^{(l-1)}_{xy}\right)^2}\left[\left(\sigma^{(l-1)}_{x}\right)^2\left(\sigma^{(l-1)}_{y}\right)^2\hat{d}^{(l-1)}_{ij}\right. \\ & - \left(\sigma^{(l-1)}_{x}\right)^2\left(\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2+\left(\sigma^{(l-1)}_{y_j\to f_{ij}}\right)^2\right)\gamma^{(l-1)}_{ij}\left(m^{(l-1)}_{x_i\to f_{ij}}-m^{(l-1)}_{x_j\to f_{ij}}\right) \\ & \left. + \left(\sigma^{(l-1)}_{y}\right)^2\left(\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2+\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2\right)\lambda^{(l-1)}_{ij}\left(m^{(l-1)}_{y_i\to f_{ij}}-m^{(l-1)}_{y_j\to f_{ij}}\right)\right], \end{aligned} \tag{4.21}$$

$$\left(\sigma^{(l)}_{f_{ij}\to\theta_i}\right)^2 = \frac{\left(\sigma^{(l-1)}_{x}\right)^2\left(\sigma^{(l-1)}_{y}\right)^2}{c^2\left(\sigma^{(l-1)}_{xy}\right)^2}, \tag{4.22}$$

where $\left(\sigma^{(l-1)}_{x}\right)^2 \triangleq \left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2 + \sigma_d^2$, $\left(\sigma^{(l-1)}_{y}\right)^2 \triangleq \left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2 + \left(\sigma^{(l-1)}_{y_j\to f_{ij}}\right)^2 + \sigma_d^2$ and $\left(\sigma^{(l-1)}_{xy}\right)^2 \triangleq \sigma_d^2 + \left(\lambda^{(l-1)}_{ij}\right)^2\left(\left(\sigma^{(l-1)}_{x_i\to f_{ij}}\right)^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2\right) + \left(\gamma^{(l-1)}_{ij}\right)^2\left(\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2 + \left(\sigma^{(l-1)}_{y_j\to f_{ij}}\right)^2\right)$.

The beliefs and the messages from variable nodes to factor nodes are calculated exactly as described in the previous sections.

However, following the same derivation process as we did in the localization problem, one would not conclude to the above expressions of the messages $\mu^{(l)}_{\xi_i\to f_{ij}}$, $\xi_i \in \{x_i, y_i, \theta_i\}$, but to the ones presented below:

$$\begin{aligned} &\mu^{(l)}_{f_{ij}\to x_i}(x_i) = \\ &\quad \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_d^2} - \frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}}\right)\right.\right. \\ &\qquad - \frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \end{aligned}$$

$$
\begin{aligned}
& - \frac{c^2 \lambda_{ij}^2 \sigma_d^4}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right)^2 \left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right)^2} \frac{\sigma_d^2 \, \sigma^2_{\theta_j \to f_{ij}}}{\left(\sigma_d^2 + c^2 \sigma^2_{\theta_j \to f_{ij}}\right)} \\
& -\mathbf{B}^2 \frac{\sigma_d^2 \, \sigma^2_{y_i \to f_{ij}}}{\sigma_d^2 + \sigma^2_{y_i \to f_{ij}}} - \mathbf{A}^2 \frac{\sigma_d^2 \, \sigma^2_{y_j \to f_{ij}}}{\sigma_d^2 + \sigma^2_{y_j \to f_{ij}}} \Bigg] \, x_i^2 \\
+ & \Bigg[\frac{\lambda_{ij} z_{ij}}{\sigma_d^2} + \frac{m_{x_j \to f_{ij}}}{\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}} - \frac{\sigma^2_{x_j \to f_{ij}}}{\sigma_d^2} \frac{\lambda_{ij} z_{ij}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right)} \\
& + \frac{c \, \lambda_{ij} \, \sigma_d^2 \, m_{\theta_i \to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right)} - \frac{c^2 \sigma^2_{\theta_i \to f_{ij}} \, \lambda_{ij} \, z_{ij}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right)} \\
& - \frac{c \, \lambda_{ij} \, \sigma_d^4 \, m_{\theta_j \to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_j \to f_{ij}}\right)} \\
& - \frac{c^2 \sigma_d^2 \sigma^2_{\theta_j \to f_{ij}} \, \lambda_{ij} \, z_{ij}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_j \to f_{ij}}\right)} \\
& + \mathbf{B} \frac{\sigma_d^2 m_{y_i \to f_{ij}}}{\sigma_d^2 + \sigma^2_{y_i \to f_{ij}}} + \mathbf{B} \frac{\sigma^2_{y_i \to f_{ij}}}{\sigma_d^2 + \sigma^2_{y_i \to f_{ij}}} (z_{ij} \, \gamma_{ij}) \\
& - \mathbf{A} \frac{\sigma_d^2 m_{y_j \to f_{ij}}}{\sigma_d^2 + \sigma^2_{y_j \to f_{ij}}} + \mathbf{A} \frac{\sigma^2_{y_j \to f_{ij}}}{\sigma_d^2 + \sigma^2_{y_j \to f_{ij}}} (z_{ij} \, \gamma_{ij}) \Bigg] \, x_i \Bigg\} ,
\end{aligned} \tag{4.23}
$$

where $\mathbf{B}$ is

$$
\mathbf{B} = \frac{\lambda_{ij} \gamma_{ij}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right)} \left(1 - \frac{\sigma_d^4}{\left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_j \to f_{ij}}\right)} \right) , \tag{4.24}
$$

and $\mathbf{A}$ is

$$
\mathbf{A} = \frac{\lambda_{ij} \gamma_{ij}}{\left(\sigma_d^2 + \sigma^2_{x_j \to f_{ij}}\right)} \left(1 - \frac{\sigma_d^4}{\left(\sigma_d^2 + c^2 \sigma^2_{\theta_i \to f_{ij}}\right) \left(\sigma_d^2 + c^2 \sigma^2_{\theta_j \to f_{ij}}\right)} \right) \frac{\sigma_d^2}{\left(\sigma_d^2 + \sigma^2_{y_i \to f_{ij}}\right)} . \tag{4.25}
$$

$$
\begin{aligned}
&\mu^{(l)}_{f_{ij}\to y_i}(y_i) = \\
&\quad \exp\Bigg\{ -\frac{1}{2}\Bigg[\frac{1}{\sigma_d^2} - \frac{\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}\right) \\
&\qquad - \frac{c^2\gamma_{ij}^2}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\gamma_{ij}^2\sigma_d^4}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad - \mathbf{B}^2\frac{\sigma_d^2\,\sigma^2_{x_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_i\to f_{ij}}} - \mathbf{A}^2\frac{\sigma_d^2\,\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}}\Bigg]\,y_i^2 \\
&\quad + \Bigg[\frac{\gamma_{ij}z_{ij}}{\sigma_d^2} + \frac{m_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}} - \frac{\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2}\frac{\gamma_{ij}z_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)} \\
&\qquad + \frac{c\,\gamma_{ij}\,\sigma_d^2\,m_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} - \frac{c^2\sigma^2_{\theta_i\to f_{ij}}\,\gamma_{ij}\,z_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \\
&\qquad - \frac{c\,\gamma_{ij}\,\sigma_d^4\,m_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\sigma_d^2\sigma^2_{\theta_j\to f_{ij}}\,\gamma_{ij}\,z_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad + \mathbf{B}\frac{\sigma_d^2 m_{x_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_i\to f_{ij}}} + \mathbf{B}\frac{\sigma^2_{x_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_i\to f_{ij}}}(z_{ij}\,\lambda_{ij}) \\
&\qquad - \mathbf{A}\frac{\sigma_d^2 m_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}} + \mathbf{A}\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}}(z_{ij}\,\lambda_{ij})\Bigg]\,y_i\Bigg\},
\end{aligned} \tag{4.26}
$$

where $\mathbf{B}$ is

$$
\mathbf{B} = \frac{\gamma_{ij}\lambda_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}\left(1 - \frac{\sigma_d^4}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right), \tag{4.27}
$$

and $\mathbf{A}$ is

$$\mathbf{A} = \frac{\gamma_{ij}\lambda_{ij}}{\left(\sigma_d^2 + \sigma_{y_j\to f_{ij}}^2\right)} \left(1 - \frac{\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma_{\theta_i\to f_{ij}}^2\right)\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)}\right) \frac{\sigma_d^2}{\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)}. \tag{4.28}$$

$$\begin{aligned}
&\mu_{\theta_i\to f_{ij}}^{(l)}(\theta_i) = \\
&\exp\Bigg\{ -\frac{1}{2}\Bigg[\frac{c^2}{\sigma_d^2} - \frac{c^2}{\sigma_d^2}\frac{c^2\sigma_{\theta_j\to f_{ij}}^2}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)} - \frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)^2}\frac{\sigma_d^2\sigma_{x_i\to f_{ij}}^2}{\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)} \\
&\qquad - \frac{c^2\gamma_{ij}^2}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)^2}\frac{\sigma_d^2\sigma_{y_i\to f_{ij}}^2}{\left(\sigma_d^2 + \sigma_{y_i\to f_{ij}}^2\right)} \\
&\qquad - \frac{c^2\lambda_{ij}^2\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)^2\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)^2}\frac{\sigma_d^2\sigma_{x_j\to f_{ij}}^2}{\left(\sigma_d^2 + \sigma_{x_j\to f_{ij}}^2\right)} \\
&\qquad - \frac{c^2\gamma_{ij}^2\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)^2\left(\sigma_d^2 + \sigma_{y_i\to f_{ij}}^2\right)^2}\frac{\sigma_d^2\sigma_{y_j\to f_{ij}}^2}{\left(\sigma_d^2 + \sigma_{y_j\to f_{ij}}^2\right)}\Bigg]\theta_i^2 \\
&\quad + \Bigg[-\frac{cz_{ij}}{\sigma_d^2} + \frac{c^2 m_{\theta_j\to f_{ij}}}{\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2} + \frac{c}{\sigma_d^2}\frac{c^2\sigma_{\theta_j\to f_{ij}}^2}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)} z_{ij} \\
&\qquad + \frac{c\lambda_{ij}\sigma_d^2 m_{x_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)} + \frac{c\lambda_{ij}^2\sigma_{x_i\to f_{ij}}^2 z_{ij}}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)} \\
&\qquad - \frac{c\lambda_{ij}\sigma_d^4 m_{x_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)\left(\sigma_d^2 + \sigma_{x_j\to f_{ij}}^2\right)} \\
&\qquad + \frac{c\lambda_{ij}^2\sigma_d^2\sigma_{x_j\to f_{ij}}^2}{\left(\sigma_d^2 + c^2\sigma_{\theta_j\to f_{ij}}^2\right)\left(\sigma_d^2 + \sigma_{x_i\to f_{ij}}^2\right)\left(\sigma_d^2 + \sigma_{x_j\to f_{ij}}^2\right)} z_{ij}
\end{aligned}$$

$$+\frac{c\gamma_{ij}\sigma_d^2 m_{y_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)}+\frac{c\gamma_{ij}^2\sigma^2_{y_i\to f_{ij}}z_{ij}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)}$$
$$-\frac{c\gamma_{ij}\sigma_d^4 m_{y_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}$$
$$\left.\left.+\frac{c\gamma_{ij}^2\sigma_d^2\sigma^2_{y_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}z_{ij}\right]\theta_i\right\}. \tag{4.29}$$

Notice that in the RHS of all three messages, $\mu^{(l)}_{f_{ij}\to x_i}$, $\mu^{(l)}_{f_{ij}\to y_i}$, $\mu^{(l)}_{f_{ij}\to \theta_i}$, we have omitted the superscripts "$(l-1)$" from the parameters λ_{ij}, γ_{ij}, $m_{\xi_j\to f_{ij}}$, $\sigma^2_{\xi_j\to f_{ij}}$, $\xi_j \in \{x_j, y_j, \theta_j\}$, to make the expressions more clear and easier to be read.

The entire derivation process of 4.23, 4.26 and 4.29 can be found in Section 7.2 of the Appendix Chapter.

Apart from the different equations for the messages from factor to variable nodes, a main difference from the work in [3] is that we have managed to simplify the Taylor expansion, which in [3] can be found in Eq. (24). The proof of the simplification is derived in Section 3.3.2 of this thesis, in Eq. 3.59, where we obtain a much simpler form of the Taylor approximation, which then makes the calculations of the BP messages easier.

Moreover, an interesting observation comes up if we consider that there is not a clock offset to estimate in the equations presented in [3], regarding both the cases of node j to be an agent or an anchor. Therefore, if we focus on the localization problem, and replace all the parameters refering to the clock offset estimation with zeros in the equations (26)-(28) and (31)-(34) in [3], i.e., put $m^{(l-1)}_{\theta_i\to f_{ij}}=0$, $m^{(l-1)}_{\theta_j\to f_{ij}}=0$ and $\left(\sigma^{(l-1)}_{\theta_i\to f_{ij}}\right)^2=0$, $\left(\sigma^{(l-1)}_{\theta_j\to f_{ij}}\right)^2=0$, we will obtain the equations 3.61, 3.62, 3.63 and 3.64 as presented in Section 3.3.2.

Chapter 5

Simulations & Results

5.1 Simulation setup

Consider a $50 \times 50\,\mathrm{m}^2$ plane with and $|\mathcal{M}| = 50$ agent nodes, uniformly distributed in space, and $|\mathcal{A}| = 9$ anchor nodes, placed in [0 0], [25 0], [50 0], [0 25], [25 25], [50 25], [0 50], [25 50] and [50 50], as shown in Fig. 5.1. The clock offset values are uniformly distributed in $[-8.3 \times 10^{-8}, 8.3 \times 10^{-8}]$. The prior distributions of agents' positions are considered to be Gaussian with mean values $m_{x_i} = m_{y_i} = 0$, and variances $\sigma^2_{x_i} = \sigma^2_{y_i} = 100\,\mathrm{m}^2$, while the priors of the clock offsets are also Gaussian with $m_{\theta_i} = 0$, $\sigma^2_{\theta_i} = 10^{-15}$, $i \in \mathcal{M}$. For the cooperative environment, the max communication range is set to 20 m. In the non-cooperative environment, we set the range to 35 m, because each agent must communicate with at least four anchor nodes, in order to estimate its clock offset and location efficiently. This is because the localization problem includes three degrees of freedom: translation, rotation and reflection, so a connection to an anchor node is needed for each degree of freedom. Here, we also need to estimate the clock offset θ, which requires one more anchor node to obtain an estimate without ambiguity. The range measurement noise is assumed to be zero mean Gaussian with variance $\sigma^2_d = 1\,\mathrm{m}^2$. The maximum number of iterations is set to $N_{iter} = 200$ and all simulation results are averaged from 100 independent Monte Carlo runs. For SPAWN [1], 4000 samples were used.

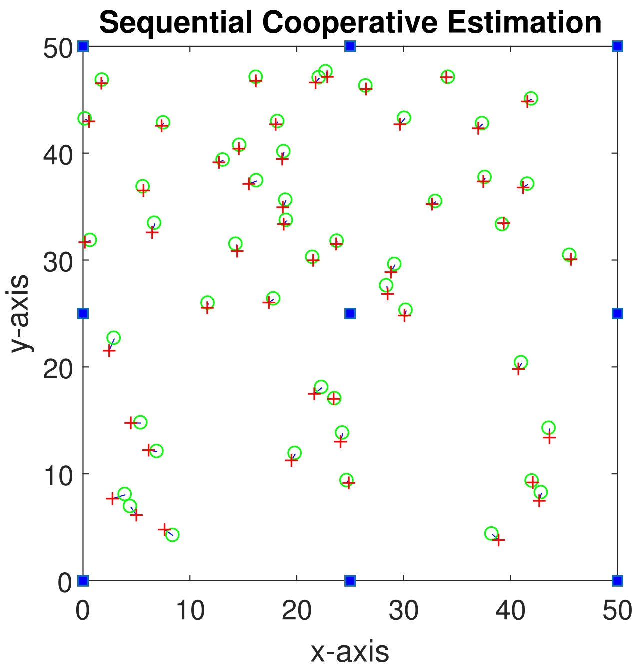

Figure 5.1: A wireless network with agents denoted by circles and anchors denoted by squares. The estimated location is expressed with the + sign.

5.2 Results & Discussion

5.2.1 Spatio-temporal estimation with zero frequency skew

Here, we present the numerical results of the algorithms described throughout the thesis, considering that the frequency skew of the local clock readings of the agent nodes is equal to zero. In the case of sequential estimation, we implement the cooperative synchronization algorithm [2] of Section 3.1 and continue with the two cooperative localization algorithms of Sections 3.3.1 and 3.3.2 [1] to compare the results. We denote the cooperative localization algorithm of Section 3.3.2 as CLBP - Cooperative Localization with Belief Propagation, and we initialize the mean values and variances of the messages $\mu^{(0)}_{\xi_i \to f_{ij}}$ to the prior mean values and variances of each variable ξ_i, with $\xi_i \in \{x_i, y_i\}, i \in \mathcal{M} \cup \mathcal{A}$. In the joint estimation case, we present the results of the non-cooperative algorithm analyzed in Section 4.1, and we denote it as JNCE - Joint Non-Cooperative Estimation. Here, we initialize the mean values and variances of the messages $\mu^{(0)}_{\xi_i \to f_{ij}}$ to zeros, with $\xi_i \in \{x_i, y_i, \theta_i\}, i \in \mathcal{M} \cup \mathcal{A}$.

The numerical results are presented below in terms of the Root Mean Square Error versus the number of Iterations that each algorithm uses in order to reach convergence. RMSE is defined as

$$RMSE = \sqrt{\frac{\sum_{i=1}^{|\mathcal{M}|} (\mathbf{x}_i - \hat{\mathbf{x}}_i)^2}{|\mathcal{M}|}} \tag{5.1}$$

where $\mathbf{x}_i$ is the real value of the parameter to be estimated, $\hat{\mathbf{x}}_i$ is the estimated parameter value and $|\mathcal{M}|$ is the total number of agent nodes that need to be synchronized and located. We evaluate the performance of thethe algorithms, both for the clock offsets and the locations, considering different noise variances (with fixed communication range) and different communication ranges (with fixed noise variance). Notice that these results are drawn using just one information packet, i.e., $N = 1$.

For the synchronization problem, in Fig. 5.2 the cooperative algorithm of Section 3.1, denoted as Coop Sync [2], converges within a few iterations (5-6), whereas JNCE needs more iterations until it reaches convergence. This can be explained by the fact that JNCE is implemented using the Taylor approximation, which has introduced parameters, such as $\lambda^{(l-1)}_{ij}, \gamma^{(l-1)}_{ij}$, that are updated at each iteration of the algorithm, alongside with the mean values and variances of the messages, and, thus, induce a delay to the convergence. Moreover, Coop Sync always performs better than JNCE regarding the accuracy of the estimation, which is expected, since it estimates only one variable, i.e., the clock offsets of the agents. Namely, Coop Sync achieves an RMSE of 6×10^{-10} sec under $1m^2$ noise variance and 2×10^{-9} sec under $10\,\mathrm{m}^2$. On the other hand, JNCE performs joint estimation, which means that synchronization is degraded by the error induced by the location estimates. This difference in the performance of Coop Sync and JNCE is more intense with higher noise variance values. JNCE demonstrates a 1.5×10^{-9} sec RMSE for

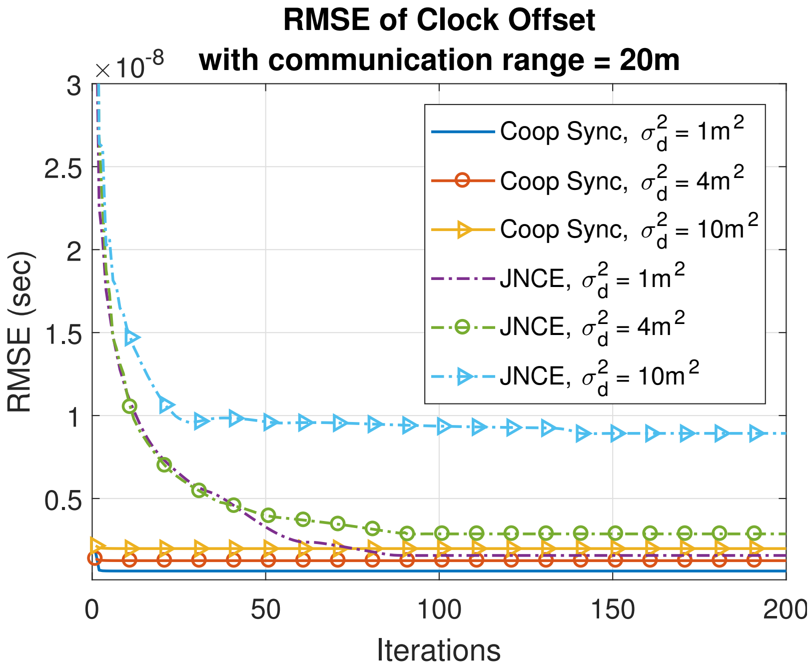

Figure 5.2: RMSE of clock offset vs. iterations, for different noise variance values

$\sigma_d^2 = 1m^2$, however, for $\sigma_d^2 = 10\,\mathrm{m}^2$, its error is huge compared to Coop Sync, i.e., $\sim 10^{-8}$, which is actually in the same order of maginitude of the real time offset values. However, the messages exchanged in the non-cooperative environment are significantly less due to the lower number of neighbors of each agent. This can be clearly seen in Table 1, where in a non-cooperative environment, an agent node has approximately 4-7 neighbors, while in the cooperative case, the number of the neighbors is raised to 8-25 for each agent. So, the number of messages to be exchanged at each iteration of the algorithm is much less for JNCE than Coop Sync, which means that the first presents lower communication requirements.

Table 5.1: Number of neighbors in Non-cooperative vs. Cooperative environment.

Environment type	Comm. range	Number of neighbors
Non-cooperative	$35m$	22 agents: 4 | 10 agents: 5 | 12 agents: 6 | 6 agents: 7
Cooperative	$20m$	15 agents: 8-14 | 22 agents: 15-20 | 13 agents: 21-25

Moving on to the localization problem, in Fig. 5.3, we observe that under the same communication range, JNCE and CLBP perform very close to SPAWN [1] after covergence. In particular, SPAWN achieves an RMSE of $\sim 0.6\,\mathrm{m}$, CLBP approximately $0.8\,\mathrm{m}$ and JNCE around $1m$, all three under the noise variance of $1\,\mathrm{m}^2$. SPAWN [1] is quite fast compared to the proposed algorithms, since it converges after 4-6 iterations, howeve, it suffers from huge communication overhead due to the fact that the messages exchanged onto the corresponding factor graph are PDFs, and need a lot of samples to be represented. For higher values of the noise variance and fixed communication range to $20\,\mathrm{m}$, the performance of CLBP and JNCE is expectedly getting worse, but remains similar between the two algorithms. The fact that CLBP and JNCE perform in a similar manner, regarding the RMSE of the final estimates, is justified, since CLBP estimates the location of the agent nodes dealing with the synchronization error added to the range measurement error, while JNCE solves the problem jointly, so it is expected that synchronization and localization are affected by each other's error throughout the whole process of the algorithm. The nature of the joint problem, though, is quite intense, and despite the fact that CLBP and JNCE behave very close to each other, CLBP always manages to outperform JNCE even if the difference is not significant.

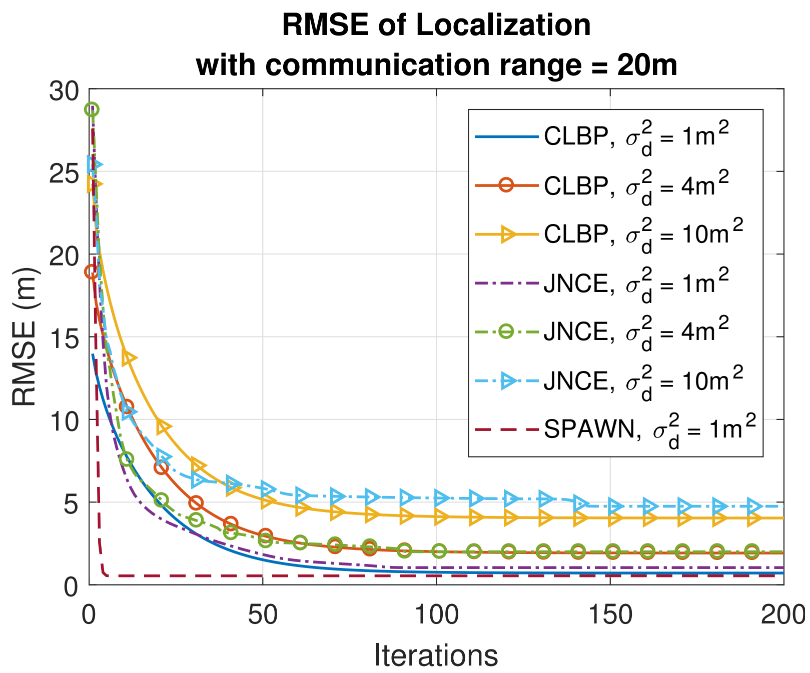

Figure 5.3: RMSE of location vs. iterations for, different noise variance values

In Fig. 5.4, we keep a fixed noise variance to 1 m^2, and present numerical results of the synchronization problem for different communication range values. Coop Sync [2] remains a quite efficient algorithm, and even under a 10 m range, it still manages to provide us with accurate enough estimates, 3×10^{-9} sec off from the real values. This can be explained by the fact that Coop Sync aims the estimation of just one variable, the clock offset θ, and an agent node needs to communicate with at least one anchor node in order to succesfully estimate its clock offset. So, a 10 m range still provides all agent nodes of the wireless network of Fig. 5.1 with more than one anchor, even if it is directly, or indirectly through some agent neighbor. An important observation comes from Fig. 5.4, regarding the JNCE algorithm, where it seems that if we increase its communication range to 70 m, the estimate RMSE remains $\sim 1.5 \times 10^{-9}$, but its convergence gets significantly faster, since it needs very few iterations (fewer than 20) to converge, compared to the case of a 35 m communication range. In a $50 \times 50\,\text{m}^2$ plane and a non-cooperative environment, a communication range of 70 m means that all

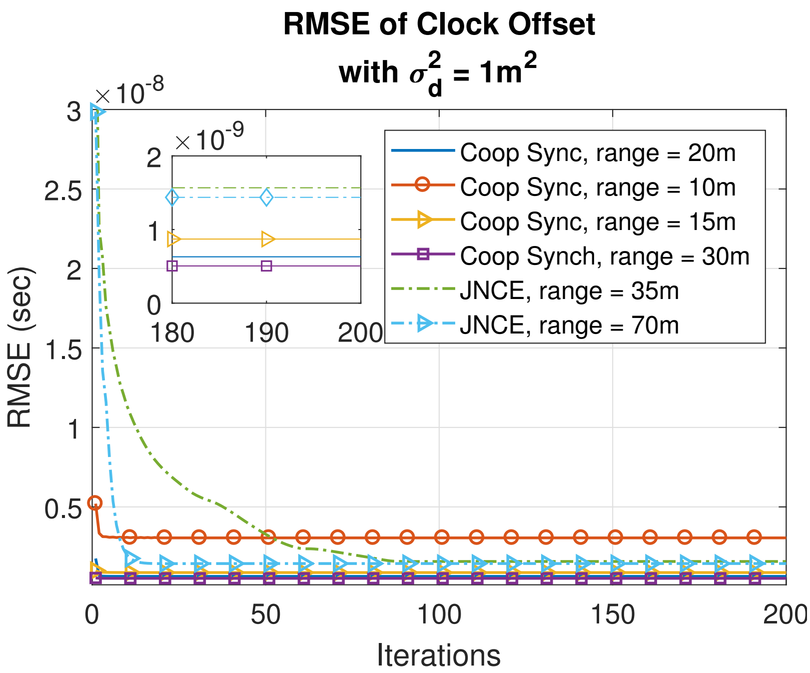

Figure 5.4: RMSE of clock offset vs. iterations, for different communication range values

agents can communicate with all anchors of the plane. It is now clear that all agents have $|\mathcal{A}| = 9$ neighbors, which remains very low compared to the 8-25 neighbors of each agent in the cooperative environment with a 20 m communication range. Also, an important aspect of JNCE is that it needs one-way measurements to perform spatio-temporal estimation, while Coop Sync [2] requires round trip timing, i.e., two-way measurements, which could be unavailable in many WSNs.

In Fig. 5.5, we evaluate the behavior of the localization algorithms for different communication range values and fixed noise variance to $1\,\mathrm{m}^2$. For CLBP, the results are expectedly better with 30 m range, but the difference with the default case of 20 m range is not that big, approximately 20 cm. On the other hand, the significance of this difference is radically increased if we reduce the communication range by 5 meters. The RMSE is then doubled, i.e., becomes 1.8 m for the 15 m range case, compared to the one of 20 m. Especially for 10 m communication range, the estimate error is unacceptably large, because it is almost 10 m. Again, the interesting part here is that JNCE

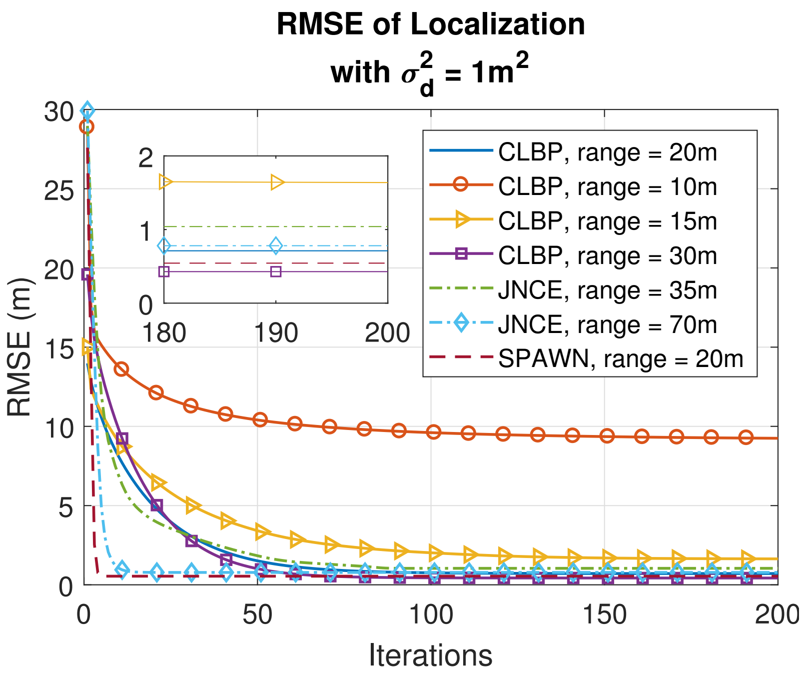

Figure 5.5: RMSE of location vs. iterations, for different communication range values

performs slighty better with 70 m communication range than 35 m (with a drop-off of 20 cm), but seriously faster. Its convergence comes after fewer than 20 iterations. For the same reasons that were explained in the previous paragraph about the clock estimation, this makes it a preferable choice over CLBP because of the significantly lower number of neighbors and, thus, the lower number of messages exchanged in order for the algorithm to converge and estimate the required parameters efficiently.

Notice that for SPAWN [1], we only present numerical results for the default setup, i.e., 1 m^2 noise variance and 20 m communication range, due to the fact that its simulations take hours to be done, as well as that the other two localization algorithms demonstrate a higher interest to evaluate further their performance.

5.2.2 Spatio-temporal estimation with non-zero frequency skew

Here, we extend the clock reading model to add the frequency offset parameter, and implement the Linear Programming method [4] of Section 3.2 to estimate the clock and frequency offset of the agent nodes. We then use the CLBP algorithm to perform network localization. Since the frequency offset requires multiple information packets to be estimated, we used $N = 20$ and $N = 100$ information packets, which means that there are N different range measurements, i.e., $\{z_{ij,n}\}_{n=1}^{N}$, which are averaged $\forall (i, j) \in \Xi$ in order to estimate the agents' location, after having estimated their clock parameters. We considered the same simulation parameters as in estimation with zero frequency offset, while the frequency skew values were uniformly generated between 20 and 40 ppm.

The Linear Programming method [4] manages to estimate the clock parameters quite efficiently. The Table 5.2 below shows the RMSE values of each clock parameter for different numbers of information packets and different values of noise variance. It seems clearly that more information packets result to more accurate estimates, while higher noise variances induce a higher error in the estimates. More particularly, it can be seen that the number of packets do not play a significant role on the clock offset estimation, since for $N = 100$ information packets the RMSE is just slightly lower than the one with $N = 20$. However, for the frequency skew, the number of packets seems to demonstrate a high impact on the estimation, since it manages to reduce the RMSE by one order of magnitude, from 10^{-7} to 10^{-8}.

Table 5.2: RMSE of the estimation of the clock parameters with the Linear Programming method. N: the number of information packets.

	RMSE of clock offset (s)		RMSE of frequency skew	
Noise variance	$N = 20$	$N = 100$	$N = 20$	$N = 100$
$\sigma_d^2 = 1m^2$	1.7633×10^{-9}	1.6670×10^{-9}	1.3053×10^{-7}	2.3920×10^{-8}
$\sigma_d^2 = 4m^2$	3.5302×10^{-9}	3.3776×10^{-9}	2.5687×10^{-7}	4.8611×10^{-8}
$\sigma_d^2 = 10m^2$	5.5155×10^{-9}	5.2282×10^{-9}	4.1570×10^{-7}	7.5123×10^{-8}

Moving on to the localization task, in Fig. 5.6, it is obvious that CLBP

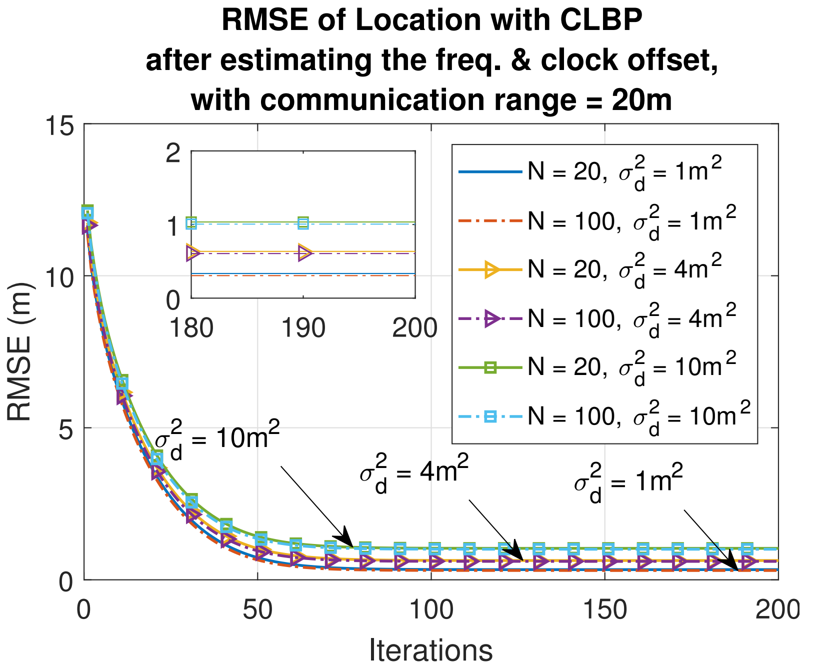

Figure 5.6: RMSE of location with CLBP vs. iterations, for the extended clock model

still performs efficiently, even with greater values of noise variance. This is because the average of the range measurements manages to limit the effect of the noise in a significant way, so that CLBP is not really affected by the induced synchronization errors or the higher noise variance. Also, it is quite obvious that the difference of the RMSE among different numbers of information packets is at least to say negligible. The important conlusion here is that CLBP manages to perform efficient localization even under more realistic modeling conditions and parameters, as well as more harsh environments.

Chapter 6

Conclusions & Future Work

This thesis aimed to solve a widely known problem that refers to the synchronization and localization in wireless networks, based on T0A and RTT measurements and utilizing message-passing techniques in a distributed way. We considered that there are anchor nodes, with perfectly known locations and synchronized to the reference time, and there are also the agent nodes that need to be synchronized and located and exchange messages with their neighbors in order to perform spatio-temporal estimation.

We first approached the problem in a sequential manner, and tried to estimate the time offsets of the agent nodes in a cooperative way, based on [4], which maps the original wireless network onto a factor graph and exploits Gaussian Belief Propagation to perform synchronization. We then extended our clock reading model and added the frequency offset parameter, which we tried to estimate with the help of the Linear Programming method. Having completed the synchronization task, we move on to the localization problem, which we solve with the help of two Belief Propagation-based algorithms: SPAWN [1], which is the go-to algorithm for localization in wireless networks, and approximate Gaussian Belief Propagation, where we used the Taylor expansions of [3] to make the computation of the GBP messages tractable.

We then attempted to jointly estimate the clock offsets and the locations of the agent nodes, exploiting similar techniques as in the sequential case. We developed a joint approximate GBP, again using that Taylor expansions [3], and evaluated its behavior both in a non-cooperative and a cooperative environment.

The simulations suggested that, regarding the synchronization problem, the algorithms that aim to solve it seperately performed better than the joint ones. As far as the localization task is concerned, SPAWN [1] managed to slightly outperform the proposed approximate GBP, but at the cost of huge communication overhead due to its sample-based implementation. Overall, the sequential approach seemed to be better than the simultaneous one, where the joint non-cooperative algorithm managed to estimate both the time offset and the location of the agent nodes efficiently, but not exactly as accurately. However, the assumption of a non-cooperative environment seriously reduces the communication requirements of the wireless network, since the anchor nodes are always much less than the agents. Regarding the joint cooperative GBP, the results showed that, under all the considerations made in this thesis, we cannot perform spatio-temporal estimation relying on it, and it is a technique that needs to be further studied and examined.

Regarding future work, the main concern that arises from this thesis is how to make the presented algorithms "smarter". The word "smarter" can refer to speed of convergence, communication requirements of the wireless network, etc. An interesting idea was to make the scheduling of the algorithms asynchronous, meaning that each node does not have to wait until **every** one of its neighbors sends their message in order to update its belief. Instead, at iteration l, each node updates its belief with all the available messages from its neighbors up to this moment. All unavailable messages can be replaced by the most recent ones. The aforementioned idea can be implemented with the generation of a probability, p_{update} that will determine if a message is updated or not at each iteration of the algorithm. More particularly, we can define a **missing ratio**, p_{miss}, so if $p_{update} > p_{miss}$, we update the current message, else we use the last updated one.

We produced several simulations to check the behavior of the algorithms under the implementation described above. In all cases where $p_{miss} > 0$, the algorithms always converged but needed more iterations than the synchronous scheduling. They usually reached the same RMSE, for different values of p_{miss}, but there were times were the RMSE would be much worse for the asynchronous scheduling, alongside with the slower convergence. Asynchronous scheduling needs further study, both in terms of sufficient convergence conditions and convergence speed, since the research field has not yet provided us with enough information on that subject.

Chapter 7

Appendix

7.1 Derivation of message $\mu^{(l)}_{f_{ij}\to x_i}(x_i)$ of Section 3.3

The derivation for the $\mu^{(l)}_{f_{ij}\to x_i}$ message is as follows:

$$
\begin{aligned}
&\mu^{(l)}_{f_{ij}\to x_i}(x_i)\\
&= \int\!\!\int\!\!\int f_{ij}(x_i, y_i, x_j, y_j)\, \mu^{(l-1)}_{x_j\to f_{ij}}(x_j)\, \mu^{(l-1)}_{y_i\to f_{ij}}(y_i)\, \mu^{(l-1)}_{y_j\to f_{ij}}(x_j)\, dx_j\, dy_i\, dy_j\\
&= \int\!\!\int\!\!\int \exp\Bigg\{ -\frac{x_i^2 - 2x_ix_j + x_j^2 + y_i^2 - 2y_iy_j + y_j^2 - 2z_{ij}\lambda^{(l-1)}_{ij}x_i}{2\sigma_d^2}\\
&\qquad\qquad \frac{+2z_{ij}\lambda^{(l-1)}_{ij}x_j - 2z_{ij}\gamma^{(l-1)}_{ij}y_i + 2z_{ij}\gamma^{(l-1)}_{ij}y_j}{2\sigma_d^2}\Bigg\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}x_j^2 + \frac{m^{(l-1)}_{x_j\to f_{ij}}}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}x_j\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}y_i^2 + \frac{m^{(l-1)}_{y_i\to f_{ij}}}{\left(\sigma^{(l-1)}_{y_i\to f_{ij}}\right)^2}y_i\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{\left(\sigma^{(l-1)}_{y_j\to f_{ij}}\right)^2}y_j^2 + \frac{m^{(l-1)}_{y_j\to f_{ij}}}{\left(\sigma^{(l-1)}_{y_j\to f_{ij}}\right)^2}y_j\right\}\, dx_j\, dy_i\, dy_j\\
&\overset{(*)}{\propto} \int \exp\left\{-\frac{x_i^2 - 2x_ix_j + x_j^2 - 2z_{ij}\lambda^{(l-1)}_{ij}x_i + 2z_{ij}\lambda^{(l-1)}_{ij}x_j}{2\sigma_d^2}\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}x_j^2 + \frac{m^{(l-1)}_{x_j\to f_{ij}}}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}x_j\right\}\, dx_j
\end{aligned}
$$

$$= \exp\left\{-\frac{x_i^2 - 2z_{ij}\lambda_{ij}^{(l-1)}x_i}{2\sigma_d^2}\right\}\int \exp\left\{-\frac{x_j^2 - 2x_ix_j + 2z_{ij}\lambda_{ij}^{(l-1)}x_j}{2\sigma_d^2}\right\}\times$$
$$\exp\left\{-\frac{1}{2}\frac{1}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}x_j^2 + \frac{m^{(l-1)}_{x_j\to f_{ij}}}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}x_j\right\}dx_j$$
$$= \exp\left\{-\frac{x_i^2 - 2z_{ij}\lambda_{ij}^{(l-1)}x_i}{2\sigma_d^2}\right\}\int \exp\left\{-\frac{1}{2}\left[\frac{1}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} + \frac{1}{\sigma_d^2}\right]x_j^2\right.$$
$$\left. + \left[\frac{m^{(l-1)}_{x_j\to f_{ij}}}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} - \frac{z_{ij}\lambda_{ij}^{(l-1)}}{\sigma_d^2}\right]x_j - \left[-\frac{1}{\sigma_d^2}\right]x_ix_j\right\}dx_j \tag{7.1}$$

where in $(*)$ we observed that the integrals of the variables y_i and y_j are independent of x_i and x_j.

To obtain the final expression of the message $\mu^{(l)}_{f_{ij}\to x_i}$, we need to solve the integral of Eq. 7.1. The lemmas below will help us and can be found in full detail in Lecture 12 of [6].

***Lemma* 1.** *If*

$$\mathbf{x} = \begin{pmatrix}\mathbf{x_1}\\ \mathbf{x_2}\end{pmatrix} \sim \mathcal{N}^{-1}\left(\begin{pmatrix}\mathbf{h_1}\\ \mathbf{h_2}\end{pmatrix}, \begin{bmatrix}\mathbf{J_{11}} & \mathbf{J_{12}}\\ \mathbf{J_{21}} & \mathbf{J_{22}}\end{bmatrix}\right) \tag{7.2}$$

then

$$\mathbf{x_1} \sim \mathcal{N}^{-1}\left(\mathbf{h}', \mathbf{J}'\right) \tag{7.3}$$

where

$$\mathbf{h}' = \mathbf{h_1} - \mathbf{J_{12}}\mathbf{J_{22}^{-1}}\mathbf{h_2} \;\; \textit{and} \;\; \mathbf{J}' = \mathbf{J_{11}} - \mathbf{J_{12}}\mathbf{J_{22}^{-1}}\mathbf{J_{21}} \tag{7.4}$$

Lemma 2. *Consider the following integral*

$$f(\mathbf{x_i}) = \int \exp\left\{ \begin{pmatrix} \mathbf{x_j} \\ \mathbf{x_i} \end{pmatrix}^T \begin{bmatrix} \mathbf{J_{jj}} & \mathbf{J_{ji}} \\ \mathbf{J_{ij}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x_j} \\ \mathbf{x_i} \end{pmatrix} + \begin{pmatrix} \mathbf{h_j} \\ \mathbf{0} \end{pmatrix}^T \begin{pmatrix} \mathbf{x_j} \\ \mathbf{x_i} \end{pmatrix} \right\} d\mathbf{x_j} \tag{7.5}$$

Applying 1, we get

$$f(\mathbf{x_i}) \propto \mathcal{N}^{-1}\left(\mathbf{x_i}; \mathbf{h}, \mathbf{J}\right) \tag{7.6}$$

where

$$\mathbf{h} = -\mathbf{J_{ij}}\mathbf{J_{jj}}^{-1}\mathbf{h_j} \quad \textit{and} \quad \mathbf{J} = -\mathbf{J_{ij}}\mathbf{J_{jj}}^{-1}\mathbf{J_{ji}} \tag{7.7}$$

Applying Lemma 2 to the integral of Eq. 7.1, we have

$$\begin{aligned} &\int \exp\left\{ -\frac{1}{2}\left[\frac{1}{(\sigma^{(l-1)}_{x_j\to f_{ij}})^2} + \frac{1}{\sigma_d^2}\right] x_j^2 + \left[\frac{m^{(l-1)}_{x_j\to f_{ij}}}{(\sigma^{(l-1)}_{x_j\to f_{ij}})^2} - \frac{z_{ij}\lambda^{(l-1)}_{ij}}{\sigma_d^2}\right] x_j \right. \\ &\qquad\qquad \left. - \left[-\frac{1}{\sigma_d^2}\right] x_i x_j \right\} dx_j \\ &\propto \mathcal{N}^{-1}\left(x_i; \mathbf{h}, \mathbf{J}\right) \end{aligned} \tag{7.8}$$

where

$$\mathbf{J} = -\left(-\frac{1}{\sigma_d^2}\right) \frac{\sigma_d^2\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}{\sigma_d^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} \left(-\frac{1}{\sigma_d^2}\right) = \frac{1}{\sigma_d^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} - \frac{1}{\sigma_d^2} \tag{7.9}$$

$$\begin{aligned} \mathbf{h} &= -\left(-\frac{1}{\sigma_d^2}\right) \frac{\sigma_d^2\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2}{\sigma_d^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} \left(\frac{m^{(l-1)}_{x_j\to f_{ij}}}{\left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} - \frac{z_{ij}\lambda^{(l-1)}_{ij}}{\sigma_d^2} \right) \\ &= \frac{m^{(l-1)}_{x_j\to f_{ij}}}{\sigma_d^2 + \left(\sigma^{(l-1)}_{x_j\to f_{ij}}\right)^2} + \left(\frac{1}{\sigma_d^2 + (\sigma^{(l-1)}_{x_j\to f_{ij}})^2} - \frac{1}{\sigma_d^2} \right) z_{ij}\lambda^{(l-1)}_{ij} \end{aligned} \tag{7.10}$$

and finally is expressed as

$$\exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}-\frac{1}{\sigma_d^2}\right)x_i^2+\left(\frac{m_{x_j\to f_{ij}}^{(l-1)}}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}\right.\right.$$
$$\left.\left.+\left(\frac{1}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}-\frac{1}{\sigma_d^2}\right)z_{ij}\lambda_{ij}^{(l-1)}\right)x_i\right\} \tag{7.11}$$

So, applying (7.11) to (7.1), the final expression is:

$$\exp\left\{-\frac{x_i^2-2z_{ij}\lambda_{ij}^{(l-1)}x_i}{2\sigma_d^2}\right\}\times$$
$$\exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}-\frac{1}{\sigma_d^2}\right)x_i^2+\left(\frac{m_{x_j\to f_{ij}}^{(l-1)}}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}\right.\right.$$
$$\left.\left.+\left(\frac{1}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}-\frac{1}{\sigma_d^2}\right)z_{ij}\lambda_{ij}^{(l-1)}\right)x_i\right\}$$
$$=\exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}\right)x_i^2+\frac{m_{x_j\to f_{ij}}^{(l-1)}+z_{ij}\lambda_{ij}^{(l-1)}}{\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2}x_i\right\}$$
$$\propto\mathcal{N}\left(x_i;m_{x_j\to f_{ij}}^{(l-1)}+z_{ij}\lambda_{ij}^{(l-1)},\sigma_d^2+\left(\sigma_{x_j\to f_{ij}}^{(l-1)}\right)^2\right) \tag{7.12}$$

7.2 Derivation of factor to variable messages for joint estimation

Here, we present the derivation of the messages $\mu_{f_{ij}\to\xi_i}^{(l)}$ of Section 4.2, with $\xi_i\in\{x_i,y_i,\theta_i\}$ and $i,j\in\mathcal{M}$, i.e., they are both agent nodes.

$$
\begin{aligned}
&\mu^{(l)}_{f_{ij}\to x_i}(x_i)\\
&= \int\int\int\int\int f_{ij}(x_i, y_i, \theta_i, x_j, y_j, \theta_j)\, \mu^{(l-1)}_{x_j\to f_{ij}}(x_j)\mu^{(l-1)}_{\theta_i\to f_{ij}}(\theta_i)\mu^{(l-1)}_{\theta_j\to f_{ij}}(\theta_j)\times\\
&\qquad \mu^{(l-1)}_{y_i\to f_{ij}}(y_i)\mu^{(l-1)}_{y_j\to f_{ij}}(y_j)dx_j d\theta_i d\theta_j dy_i dy_j\\
&\propto \int\int\int\int\int \exp\left\{-\frac{1}{2}\frac{1}{(\sigma^{(l-1)}_{x_j\to f_{ij}})^2}\, x_j^2 + \frac{m^{(l-1)}_{x_j\to f_{ij}}}{(\sigma^{(l-1)}_{x_j\to f_{ij}})^2}\, x_j\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma^{(l-1)}_{\theta_i\to f_{ij}})^2}\, \theta_i^2 + \frac{m^{(l-1)}_{\theta_i\to f_{ij}}}{(\sigma^{(l-1)}_{\theta_i\to f_{ij}})^2}\, \theta_i\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma^{(l-1)}_{\theta_j\to f_{ij}})^2}\, \theta_j^2 + \frac{m^{(l-1)}_{\theta_j\to f_{ij}}}{(\sigma^{(l-1)}_{\theta_j\to f_{ij}})^2}\, \theta_j\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma^{(l-1)}_{y_i\to f_{ij}})^2}\, y_i^2 + \frac{m^{(l-1)}_{y_i\to f_{ij}}}{(\sigma^{(l-1)}_{y_i\to f_{ij}})^2}\, y_i\right\}\times\\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma^{(l-1)}_{y_j\to f_{ij}})^2}\, y_j^2 + \frac{m^{(l-1)}_{y_j\to f_{ij}}}{(\sigma^{(l-1)}_{y_j\to f_{ij}})^2}\, y_j\right\}\times\\
&\qquad \exp\left\{-\frac{z_{ij}^2 + x_i^2 - 2x_i x_j + x_j^2 + y_i^2 - 2y_i y_j + y_j^2 + c^2\theta_j^2}{2\sigma_d^2}\right.\\
&\qquad\qquad \frac{-2c^2\theta_j\theta_i + c^2\theta_i^2 - 2z_{ij}c\theta_j + 2z_{ij}c\theta_i - 2z_{ij}\lambda^{(l-1)}_{ij}x_i}{2\sigma_d^2}\\
&\qquad\qquad \frac{+2z_{ij}\lambda^{(l-1)}_{ij}x_j - 2z_{ij}\gamma^{(l-1)}_{ij}y_i + 2z_{ij}\gamma^{(l-1)}_{ij}y_j}{2\sigma_d^2}\\
&\qquad\qquad \frac{+2c\theta_j\lambda^{(l-1)}_{ij}x_i - 2c\theta_j\lambda^{(l-1)}_{ij}x_j + 2c\theta_j\gamma^{(l-1)}_{ij}y_i}{2\sigma_d^2}\\
&\qquad\qquad \frac{-2c\theta_j\gamma^{(l-1)}_{ij}y_j - 2c\theta_i\lambda^{(l-1)}_{ij}x_i + 2c\theta_i\lambda^{(l-1)}_{ij}x_j}{2\sigma_d^2}\\
&\qquad\qquad \left.\frac{-2c\theta_i\gamma^{(l-1)}_{ij}y_i + 2c\theta_i\gamma^{(l-1)}_{ij}y_j}{2\sigma_d^2}\right\} dx_j d\theta_i d\theta_j dy_i dy_j\\
&\overset{(*)}{\propto} \exp\left\{-\frac{x_i^2 - 2z_{ij}\lambda_{ij}x_i}{2\sigma_d^2}\right\}\times\\
&\quad \int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_{ij}}}y_j^2 + \frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2 + 2z_{ij}\gamma_{ij}y_j}{2\sigma_d^2}\right\}\times
\end{aligned}
$$

$$
\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2 + \frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\}\exp\left\{-\frac{y_i^2 - 2y_iy_j - 2z_{ij}\gamma_{ij}y_i}{2\sigma_d^2}\right\}\times \\
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j^2 + \frac{m_{\theta_j\to f_{ij}}}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j\right\}\times \\
&\quad \exp\left\{-\frac{c^2\theta_j^2 - 2z_{ij}c\theta_j + 2c\lambda_{ij}\theta_jx_i + 2c\gamma_{ij}\theta_jy_i - 2c\gamma_{ij}\theta_jy_j}{2\sigma_d^2}\right\}\times \\
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_i\to f_{ij}}}\theta_i^2 + \frac{m_{\theta_i\to f_{ij}}}{\sigma^2_{\theta_i\to f_{ij}}}\theta_i\right\}\times \\
&\quad \exp\left\{-\frac{c^2\theta_i^2 - 2c^2\theta_j\theta_i + 2z_{ij}c\theta_i - 2c\lambda_{ij}\theta_ix_i - 2c\gamma_{ij}\theta_iy_i + 2c\gamma_{ij}\theta_iy_j}{2\sigma_d^2}\right\}\times \\
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_j\to f_{ij}}}x_j^2 + \frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}}x_j\right\}\times \\
&\quad \exp\left\{-\frac{x_j^2 - 2x_ix_j + 2z_{ij}\lambda_{ij}x_j + 2c\lambda_{ij}\theta_ix_j - 2c\lambda_{ij}\theta_jx_j}{2\sigma_d^2}\right\}dx_jd\theta_id\theta_jdy_idy_j,
\end{aligned}
\tag{7.13}
$$

where in $(*)$ we have omitted the supescripts "$(l-1)$" in to make the derivation easier for the reader. Eq 7.13 consists of 5 nested integrals and the process that we need to follow to get to the solution is now clear. We start by calculating the inner integral according to the variable x_j. Again, we deal with those integrals as we did in chapter 7.1 of the Appendix. So using Lemma 2 for the last nested integral according to variable x_j, we obtain:

$$
\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_j\to f_{ij}}}x_j^2+\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}}x_j\right\}\times \\
&\quad \exp\left\{-\frac{x_j^2-2x_ix_j+2z_{ij}\lambda_{ij}x_j+2c\lambda_{ij}\theta_ix_j-2c\lambda_{ij}\theta_jx_j}{2\sigma_d^2}\right\}dx_j \\
&=\int \exp\Bigg\{-\frac{1}{2}\left[\frac{1}{\sigma^2_{x_j\to f_{ij}}}+\frac{1}{\sigma_d^2}\right]x_j^2+\left[\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}}-\frac{z_{ij}\lambda_{ij}+c\lambda_{ij}\theta_i-c\lambda_{ij}\theta_j}{\sigma_d^2}\right]x_j \\
&\qquad -\left[-\frac{1}{\sigma_d^2}\right]x_jx_i\Bigg\}dx_j \\
&\propto \exp\Bigg\{-\frac{1}{2}\left[-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)\right]x_i^2+\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}x_i \\
&\qquad -\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)(z_{ij}\lambda_{ij}+c\lambda_{ij}\theta_i-c\lambda_{ij}\theta_j)\,x_i\Bigg\}
\end{aligned}
\tag{7.14}
$$

So, using (7.14) in (7.13), we have:

$$
\begin{aligned}
&\exp\left\{-\frac{x_i^2-2z_{ij}\lambda_{ij}x_i}{2\sigma_d^2}\right\}\exp\Bigg\{-\frac{1}{2}\left[-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)\right]x_i^2\\
&\qquad+\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\,x_i-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)z_{ij}\lambda_{ij}\,x_i\Bigg\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_{ij}}}y_j^2+\frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}\gamma_{ij}y_j}{2\sigma_d^2}\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2+\frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\}\exp\left\{-\frac{y_i^2-2y_iy_j-2z_{ij}\gamma_{ij}y_i}{2\sigma_d^2}\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j^2+\frac{m_{\theta_j\to f_{ij}}}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j\right\}\times\\
&\quad\exp\left\{-\frac{c^2\theta_j^2-2z_{ij}c\theta_j+2c\lambda_{ij}\theta_jx_i+2c\gamma_{ij}\theta_jy_i-2c\gamma_{ij}\theta_jy_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)c\lambda_{ij}\theta_jx_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_i\to f_{ij}}}\theta_i^2+\frac{m_{\theta_i\to f_{ij}}}{\sigma^2_{\theta_i\to f_{ij}}}\theta_i\right\}\times\\
&\quad\exp\left\{-\frac{c^2\theta_i^2-2c^2\theta_j\theta_i+2z_{ij}c\theta_i-2c\lambda_{ij}\theta_ix_i-2c\gamma_{ij}\theta_iy_i+2c\gamma_{ij}\theta_iy_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)c\lambda_{ij}\theta_ix_i\right\}d\theta_id\theta_jdy_idy_j
\end{aligned}
\tag{7.15}
$$

We procceed with the solution of the current inner integral according to variable θ_i:

$$
\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_i\to f_{ij}}}\theta_i^2 + \frac{m_{\theta_i\to f_{ij}}}{\sigma^2_{\theta_i\to f_{ij}}}\theta_i\right\}\times \\
&\quad \exp\left\{-\frac{c^2\theta_i^2 - 2c^2\theta_j\theta_i + 2z_{ij}c\theta_i - 2c\lambda_{ij}\theta_i x_i - 2c\gamma_{ij}\theta_i y_i + 2c\gamma_{ij}\theta_i y_j}{2\sigma_d^2}\right\}\times \\
&\quad \exp\left\{-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)c\lambda_{ij}\theta_i x_i\right\}d\theta_i \\
= &\int \exp\Bigg\{-\frac{1}{2}\left[\frac{1}{\sigma^2_{\theta_i\to f_{ij}}}+\frac{c^2}{\sigma_d^2}\right]\theta_i + \left[\frac{m_{\theta_i\to f_{ij}}}{\sigma^2_{\theta_i\to f_{ij}}}+\frac{c}{\sigma_d^2}\left(c\theta_j - z_{ij}+\gamma_{ij}y_i-\gamma_{ij}y_j\right)\right]\theta_i \\
&\qquad -\left[\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)c\lambda_{ij}-\frac{c\lambda_{ij}}{\sigma_d^2}\right]\theta_i\, x_i\Bigg\}d\theta_i \\
\propto &\exp\Bigg\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^2\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\right]x_i^2 \\
&\qquad + \frac{c\lambda_{ij}\,\sigma_d^2\,m_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}x_i \\
&\qquad + \frac{c^2\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\left(c\theta_j - z_{ij}+\gamma_{ij}y_i-\gamma_{ij}y_j\right)x_i\Bigg\}
\end{aligned}
\tag{7.16}
$$

Applying 7.16 into 7.15, the latter now becomes:

$$
\begin{aligned}
&\exp\left\{-\frac{x_i^2 - 2z_{ij}\lambda_{ij}x_i}{2\sigma_d^2}\right\}\exp\left\{-\frac{1}{2}\left[-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)\right]x_i^2\right.\\
&\qquad\left. + \frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\,x_i - \frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)z_{ij}\lambda_{ij}\,x_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^2\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\right]x_i^2\right.\\
&\left.+\frac{c\lambda_{ij}\,\sigma_d^2\,m_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,x_i - \frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}z_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}x_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_{ij}}}y_j^2+\frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}\gamma_{ij}y_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{-\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,y_jx_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2+\frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\}\exp\left\{-\frac{y_i^2-2y_iy_j-2z_{ij}\gamma_{ij}y_i}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,y_i\,x_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j^2+\frac{m_{\theta_j\to f_{ij}}}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j\right\}\times\\
&\quad\exp\left\{-\frac{c^2\theta_j^2-2z_{ij}c\theta_j+2c\lambda_{ij}\theta_jx_i+2c\gamma_{ij}\theta_jy_i-2c\gamma_{ij}\theta_jy_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)c\lambda_{ij}\theta_jx_i\right\}\times\\
&\quad\exp\left\{\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,c\,\theta_j\,x_i\right\}d\theta_jdy_idy_j
\end{aligned}
\tag{7.17}
$$

To save time and space, we will now give only the solution of each integral, since we showed how to calculate them. The solution of the current inner integral according to the variable θ_j is:

$$\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j^2+\frac{m_{\theta_j\to f_{ij}}}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j\right\}\times\\
&\quad\exp\left\{-\frac{c^2\theta_j^2-2z_{ij}c\theta_j+2c\lambda_{ij}\theta_j x_i+2c\gamma_{ij}\theta_j y_i-2c\gamma_{ij}\theta_j y_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)c\lambda_{ij}\theta_j x_i\right\}\times\\
&\quad\exp\left\{\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,c\,\theta_j\,x_i\right\}d\theta_j\\
&=\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^4}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right]x_i^2\right.\\
&\qquad-\frac{c\,\lambda_{ij}\,\sigma_d^4\,m_{\theta_j\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}x_i\\
&\quad\left.-\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\left(z_{ij}-\gamma_{ij}y_i+\gamma_{ij}y_j\right)x_i\right\}
\end{aligned}\tag{7.18}$$

Using 7.18 into 7.17, we obtain:

$$\exp\left\{-\frac{x_i^2 - 2z_{ij}\lambda_{ij}x_i}{2\sigma_d^2}\right\}\exp\left\{ -\frac{1}{2}\left[-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)\right] x_i^2 \right.$$
$$\left. + \frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\, x_i - \frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right) z_{ij}\lambda_{ij}\, x_i\right\}\times$$
$$\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^2\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\right] x_i^2\right.$$
$$\left. +\frac{c\lambda_{ij}\,\sigma_d^2\, m_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\, x_i - \frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}} z_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} x_i\right\}\times$$
$$\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^4}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right] x_i^2\right.$$
$$-\frac{c\,\lambda_{ij}\,\sigma_d^4\, m_{\theta_j\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\, x_i$$
$$\left.-\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}\, z_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\, x_i\right\}$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_i j}} y_j^2 + \frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}} y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}\gamma_{ij}y_j}{2\sigma_d^2}\right\}\times$$
$$\exp\left\{-\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\, y_j x_i\right\}\times$$
$$\exp\left\{-\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}\,\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}(y_j\, x_i)\right\}\times$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}} y_i^2 + \frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}} y_i\right\}\exp\left\{-\frac{y_i^2-2y_iy_j-2z_{ij}\gamma_{ij}y_i}{2\sigma_d^2}\right\}\times$$
$$\exp\left\{\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}\,\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\, y_i\, x_i\right\}\times$$

$$\exp\left\{\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma_{\theta_j\to f_{ij}}^2\,\gamma_{ij}}{\left(\sigma_{x_j\to f_{ij}}^2+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma_{\theta_i\to f_{ij}}^2\right)\left(\sigma_d^2+c^2\sigma_{\theta_j\to f_{ij}}^2\right)}(y_i\,x_i)\right\}dy_i dy_j \tag{7.19}$$

The inner integral according to y_i is solved as:

$$\exp\left\{-\frac{1}{2}\,(-\mathbf{B}^2)\,\frac{\sigma_d^2\,\sigma_{y_i\to f_{ij}}^2}{\sigma_d^2+\sigma_{y_i\to f_{ij}}^2}\,x_i^2+\mathbf{B}\,\frac{\sigma_d^2\,m_{y_i\to f_{ij}}}{\sigma_d^2+\sigma_{y_i\to f_{ij}}^2}\,x_i + \mathbf{B}\,\frac{\sigma_{y_i\to f_{ij}}^2}{\sigma_d^2+\sigma_{y_i\to f_{ij}}^2}(y_j+z_{ij}\gamma_{ij})\,x_i\right\} \tag{7.20}$$

where $\mathbf{B}$ is

$$\mathbf{B}=\frac{\lambda_{ij}\gamma_{ij}}{\left(\sigma_d^2+\sigma_{x_j\to f_{ij}}^2\right)}\left(1-\frac{\sigma_d^4}{\left(\sigma_d^2+c^2\sigma_{\theta_i\to f_{ij}}^2\right)\left(\sigma_d^2+c^2\sigma_{\theta_j\to f_{ij}}^2\right)}\right) \tag{7.21}$$

Putting 7.20 into 7.19, the second is now:

$$
\begin{aligned}
&\exp\left\{-\frac{x_i^2 - 2z_{ij}\lambda_{ij}x_i}{2\sigma_d^2}\right\}\exp\left\{-\frac{1}{2}\left[-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)\right]x_i^2\right.\\
&\qquad\left.+\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\,x_i-\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma^2_{x_j\to f_{ij}}+\sigma_d^2}\right)z_{ij}\lambda_{ij}\,x_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^2\,\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\right]x_i^2\right.\\
&\quad\left.+\frac{c\lambda_{ij}\,\sigma_d^2\,m_{\theta_i\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,x_i-\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}z_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}x_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\,\sigma_d^4}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right]x_i^2\right.\\
&\qquad-\frac{c\,\lambda_{ij}\,\sigma_d^4\,m_{\theta_j\to f_{ij}}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,x_i\\
&\qquad\left.-\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}\,z_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,x_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\,(-\mathbf{B})^2\,\frac{\sigma_d^2\,\sigma^2_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}}\,x_i^2+\mathbf{B}\,\frac{\sigma_d^2\,m_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}}\,x_i+\mathbf{B}\,\frac{\sigma^2_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}}(z_{ij}\gamma_{ij})\,x_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_ij}}y_j^2+\frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}\gamma_{ij}y_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{-\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,y_jx_i\right\}\times\\
&\quad\exp\left\{-\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}\,\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}y_j\,x_i\right\}\times\\
&\quad\exp\left\{\mathbf{B}\,\frac{\sigma^2_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}}y_j\,x_i\right\}dy_j
\end{aligned}
\tag{7.22}
$$

Finally, the last integral according to y_j is solved as:

$$\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_ij}}y_j^2 + \frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}\gamma_{ij}y_j}{2\sigma_d^2}\right\}\times\\
&\exp\left\{-\frac{c^2\lambda_{ij}\,\sigma^2_{\theta_i\to f_{ij}}\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)}\,y_jx_i\right\}\times\\
&\exp\left\{-\frac{c^2\,\lambda_{ij}\,\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}\,\gamma_{ij}}{\left(\sigma^2_{x_j\to f_{ij}}+\sigma_d^2\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}y_j\,x_i\right\}\times\\
&\exp\left\{\mathbf{B}\,\frac{\sigma^2_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}}y_j\,x_i\right\}dy_j\\
&\propto\exp\left\{-\frac{1}{2}(-\mathbf{A}^2)\frac{\sigma_d^2\,\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}\,x_i^2-\mathbf{A}\,\frac{\sigma_d^2\,m_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}\,x_i+\mathbf{A}\,\frac{\sigma^2_{y_j\to f_{ij}}\gamma_{ij}z_{ij}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}\,x_i\right\}
\end{aligned} \tag{7.23}$$

where $\mathbf{A}$ is

$$\mathbf{A}=\frac{\lambda_{ij}\gamma_{ij}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)}\left(1-\frac{\sigma_d^4}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right)\frac{\sigma_d^2}{\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)} \tag{7.24}$$

So, the final expression for the message $\mu^{(l)}_{f_{ij}\to x_i}$ is:

$$
\begin{aligned}
&\mu^{(l)}_{f_{ij}\to x_i}(x_i) = \\
&\quad \exp\Bigg\{ -\frac{1}{2}\Bigg[\frac{1}{\sigma_d^2} - \frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}}\right) \\
&\qquad\quad - \frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \\
&\qquad\quad - \frac{c^2\lambda_{ij}^2\sigma_d^4}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad\quad - \mathbf{B}^2\frac{\sigma_d^2\,\sigma^2_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}} - \mathbf{A}^2\frac{\sigma_d^2\,\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}\Bigg]x_i^2 \\
&\quad + \Bigg[\frac{\lambda_{ij}z_{ij}}{\sigma_d^2} + \frac{m_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}} - \frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2}\frac{\lambda_{ij}z_{ij}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)} \\
&\qquad + \frac{c\,\lambda_{ij}\,\sigma_d^2\,m_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} - \frac{c^2\sigma^2_{\theta_i\to f_{ij}}\,\lambda_{ij}\,z_{ij}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \\
&\qquad - \frac{c\,\lambda_{ij}\,\sigma_d^4\,m_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\sigma_d^2\sigma^2_{\theta_j\to f_{ij}}\,\lambda_{ij}\,z_{ij}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad + \mathbf{B}\frac{\sigma_d^2 m_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}} + \mathbf{B}\frac{\sigma^2_{y_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_i\to f_{ij}}}(z_{ij}\,\gamma_{ij}) \\
&\qquad - \mathbf{A}\frac{\sigma_d^2 m_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}} + \mathbf{A}\frac{\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}(z_{ij}\,\gamma_{ij})\Bigg]x_i\Bigg\}
\end{aligned}
\tag{7.25}
$$

The derivation for the message $\mu^{(l)}_{f_{ij}\to y_i}$ is omitted, because it is entirely symmetrical with the one of $\mu^{(l)}_{f_{ij}\to x_i}$. The final expression for the message

$\mu^{(l)}_{f_{ij}\to y_i}$ is:

$$
\begin{aligned}
&\mu^{(l)}_{f_{ij}\to y_i}(y_i) = \\
&\quad \exp\Bigg\{ -\frac{1}{2}\Bigg[\frac{1}{\sigma_d^2} - \frac{\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2}\left(\frac{1}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}}\right) \\
&\qquad - \frac{c^2\gamma_{ij}^2}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\gamma_{ij}^2\sigma_d^4}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)^2\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\,\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad - \mathbf{B}^2\frac{\sigma_d^2\,\sigma^2_{x_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_i\to f_{ij}}} - \mathbf{A}^2\frac{\sigma_d^2\,\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}}\Bigg] y_i^2 \\
&\quad + \Bigg[\frac{\gamma_{ij}z_{ij}}{\sigma_d^2} + \frac{m_{y_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{y_j\to f_{ij}}} - \frac{\sigma^2_{y_j\to f_{ij}}}{\sigma_d^2}\frac{\gamma_{ij}z_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)} \\
&\qquad + \frac{c\,\gamma_{ij}\,\sigma_d^2\,m_{\theta_i\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} - \frac{c^2\sigma^2_{\theta_i\to f_{ij}}\,\gamma_{ij}\,z_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)} \\
&\qquad - \frac{c\,\gamma_{ij}\,\sigma_d^4\,m_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\sigma_d^2\sigma^2_{\theta_j\to f_{ij}}\,\gamma_{ij}\,z_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)} \\
&\qquad + \mathbf{B}\frac{\sigma_d^2 m_{x_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_i\to f_{ij}}} + \mathbf{B}\frac{\sigma^2_{x_i\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_i\to f_{ij}}}(z_{ij}\,\lambda_{ij}) \\
&\qquad - \mathbf{A}\frac{\sigma_d^2 m_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}} + \mathbf{A}\frac{\sigma^2_{x_j\to f_{ij}}}{\sigma_d^2+\sigma^2_{x_j\to f_{ij}}}(z_{ij}\,\lambda_{ij})\Bigg] y_i\Bigg\}
\end{aligned} \tag{7.26}
$$

where $\mathbf{B}$ is

$$
\mathbf{B} = \frac{\gamma_{ij}\lambda_{ij}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}\left(1-\frac{\sigma_d^4}{\left(\sigma_d^2+c^2\sigma^2_{\theta_i\to f_{ij}}\right)\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right) \tag{7.27}
$$

and $\mathbf{A}$ is

$$\mathbf{A} = \frac{\gamma_{ij}\lambda_{ij}}{\left(\sigma_d^2 + \sigma_{y_j \to f_{ij}}^2\right)} \left(1 - \frac{\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma_{\theta_i \to f_{ij}}^2\right)\left(\sigma_d^2 + c^2\sigma_{\theta_j \to f_{ij}}^2\right)}\right) \frac{\sigma_d^2}{\left(\sigma_d^2 + \sigma_{x_i \to f_{ij}}^2\right)} \tag{7.28}$$

We now move on with the derviation $\mu_{f_{ij}\to\theta_i}^{(l)}(\theta_i)$, following exactly the same method as shown above.

$$\begin{aligned}
&\mu_{f_{ij}\to\theta_i}^{(l)}(\theta_i) \\
&= \int\int\int\int\int f_{ij}(x_i, y_i, \theta_i, x_j, y_j, \theta_j)\, \mu_{\theta_j \to f_{ij}}^{(l-1)}(\theta_j)\mu_{x_i \to f_{ij}}^{(l-1)}(x_i)\mu_{x_j \to f_{ij}}^{(l-1)}(x_j)\times \\
&\qquad\qquad \mu_{y_i \to f_{ij}}^{(l-1)}(y_i)\mu_{y_j \to f_{ij}}^{(l-1)}(y_j)d\theta_j dx_i dx_j dy_i dy_j \\
&\propto \int\int\int\int\int \exp\left\{-\frac{1}{2}\frac{1}{(\sigma_{\theta_j \to f_{ij}}^{(l-1)})^2}\,\theta_j^2 + \frac{m_{\theta_j \to f_{ij}}^{(l-1)}}{(\sigma_{\theta_j \to f_{ij}}^{(l-1)})^2}\,\theta_j\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma_{x_i \to f_{ij}}^{(l-1)})^2}\,x_i^2 + \frac{m_{x_i \to f_{ij}}^{(l-1)}}{(\sigma_{x_i \to f_{ij}}^{(l-1)})^2}\,x_i\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma_{x_j \to f_{ij}}^{(l-1)})^2}\,x_j^2 + \frac{m_{x_j \to f_{ij}}^{(l-1)}}{(\sigma_{x_j \to f_{ij}}^{(l-1)})^2}\,x_j\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma_{y_i \to f_{ij}}^{(l-1)})^2}\,y_i^2 + \frac{m_{y_i \to f_{ij}}^{(l-1)}}{(\sigma_{y_i \to f_{ij}}^{(l-1)})^2}\,y_i\right\}\times \\
&\qquad \exp\left\{-\frac{1}{2}\frac{1}{(\sigma_{y_j \to f_{ij}}^{(l-1)})^2}\,y_j^2 + \frac{m_{y_j \to f_{ij}}^{(l-1)}}{(\sigma_{y_j \to f_{ij}}^{(l-1)})^2}\,y_j\right\}\times \\
&\qquad \exp\left\{-\frac{z_{ij}^2 + x_i^2 - 2x_ix_j + x_j^2 + y_i^2 - 2y_iy_j + y_j^2 + c^2\theta_j^2}{2\sigma_d^2}\right. \\
&\qquad\quad \frac{-2c^2\theta_j\theta_i + c^2\theta_i^2 - 2z_{ij}c\theta_j + 2z_{ij}c\theta_i - 2z_{ij}\lambda_{ij}^{(l-1)}x_i}{2\sigma_d^2} \\
&\qquad\quad \frac{+2z_{ij}\lambda_{ij}^{(l-1)}x_j - 2z_{ij}\gamma_{ij}^{(l-1)}y_i + 2z_{ij}\gamma_{ij}^{(l-1)}y_j}{2\sigma_d^2}
\end{aligned}$$

$$\frac{+2c\theta_j\lambda_{ij}^{(l-1)}x_i - 2c\theta_j\lambda_{ij}^{(l-1)}x_j + 2c\theta_j\gamma_{ij}^{(l-1)}y_i}{2\sigma_d^2}$$
$$\frac{-2c\theta_j\gamma_{ij}^{(l-1)}y_j - 2c\theta_i\lambda_{ij}^{(l-1)}x_i + 2c\theta_i\lambda_{ij}^{(l-1)}x_j}{2\sigma_d^2}$$
$$\left.\frac{-2c\theta_i\gamma_{ij}^{(l-1)}y_i + 2c\theta_i\gamma_{ij}^{(l-1)}y_j}{2\sigma_d^2}\right\}d\theta_j dx_i dx_j dy_i dy_j$$
$$\propto \exp\left\{-\frac{c^2\theta_i^2 + 2z_{ij}\,c\,\theta_i}{2\sigma_d^2}\right\}\times$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_ij}}y_j^2 + \frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2 + 2z_{ij}\gamma_{ij}y_j + 2c\gamma_{ij}\theta_i y_i}{2\sigma_d^2}\right\}\times$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2 + \frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\}\times$$
$$\exp\left\{-\frac{y_i^2 - 2y_iy_j - 2z_{ij}\gamma_{ij}y_i - 2c\gamma_{ij}\theta_i y_i}{2\sigma_d^2}\right\}\times$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_j\to f_{ij}}}x_j^2 + \frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}}x_j\right\}\exp\left\{-\frac{x_j^2 + 2z_{ij}\lambda_{ij}x_j + 2c\lambda_{ij}\theta_i x_j}{2\sigma_d^2}\right\}\times$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_i\to f_{ij}}}x_i^2 + \frac{m_{\theta_i\to f_{ij}}}{\sigma^2_{x_i\to f_{ij}}}x_i\right\}\times$$
$$\exp\left\{-\frac{x_i^2 - 2x_jx_i - 2z_{ij}\lambda_{ij}x_i - 2c\lambda_{ij}\theta_i x_i}{2\sigma_d^2}\right\}\times$$
$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j^2 + \frac{m_{\theta_j\to f_{ij}}}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j\right\}\times$$
$$\exp\left\{-\frac{c^2\theta_j^2 - 2c^2\theta_i\theta_j - 2z_{ij}c\theta_j + 2c\lambda_{ij}\theta_j x_i - 2c\lambda_{ij}\theta_j x_j + 2c\gamma_{ij}\theta_j y_i}{2\sigma_d^2}\right.$$
$$\left.\frac{-2c\gamma_{ij}\theta_j y_j}{2\sigma_d^2}\right\}d\theta_j dx_i dx_j dy_i dy_j \tag{7.29}$$

The inner integral according to the variable θ_j is solved as:

$$\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j^2+\frac{m_{\theta_j\to f_{ij}}}{\sigma^2_{\theta_j\to f_{ij}}}\theta_j\right\}\exp\left\{-\frac{c^2\theta_j^2-2c^2\theta_i\theta_j-2z_{ij}c\theta_j}{2\sigma_d^2}\right.\\
&\qquad\left.\frac{+2c\lambda_{ij}\theta_j x_i-2c\lambda_{ij}\theta_j x_j+2c\gamma_{ij}\theta_j y_i-2c\gamma_{ij}\theta_j y_j}{2\sigma_d^2}\right\}d\theta_j\\
&\propto \exp\left\{-\frac{1}{2}\left[-\frac{c^2}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right]\theta_i^2+\frac{c^2 m_{\theta_j\to f_{ij}}}{\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}}\theta_i\right.\\
&\qquad\left.+\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\left(z_{ij}-\lambda_{ij}x_i+\lambda_{ij}x_j-\gamma_{ij}y_i+\gamma_{ij}y_j\right)\theta_i\right\}
\end{aligned} \tag{7.30}$$

Putting 7.30 into 7.29 we obtain:

$$
\begin{aligned}
&\exp\left\{-\frac{c^2\theta_i^2 + 2z_{ij}\, c\, \theta_i}{2\sigma_d^2}\right\} \times \\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right]\theta_i^2 + \frac{c^2 m_{\theta_j\to f_{ij}}}{\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}}\,\theta_i \right.\\
&\qquad\qquad \left. + \frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)} z_{ij}\,\theta_i\right\} \times \\
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_i j}} y_j^2 + \frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}} y_j\right\} \exp\left\{-\frac{y_j^2 + 2z_{ij}\gamma_{ij}y_j + 2c\gamma_{ij}\theta_i y_i}{2\sigma_d^2}\right\} \times \\
&\quad \exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\gamma_{ij}\, y_j\, \theta_i\right\} \times \\
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}} y_i^2 + \frac{m_{y_i\to f_{ij}} - 2c\gamma_{ij}\theta_i y_i}{\sigma^2_{y_i\to f_{ij}}} y_i\right\} \exp\left\{-\frac{y_i^2 - 2y_i y_j - 2z_{ij}\gamma_{ij} y_i}{2\sigma_d^2}\right\} \times \\
&\quad \exp\left\{-\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\gamma_{ij}\, y_i\, \theta_i\right\} \times \\
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_j\to f_{ij}}} x_j^2 + \frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}} x_j\right\} \exp\left\{-\frac{x_j^2 + 2z_{ij}c\lambda_{ij}x_j + 2c\lambda_{ij}\theta_i x_j}{2\sigma_d^2}\right\} \times \\
&\quad \exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\lambda_{ij}\, x_j\, \theta_i\right\} dx_j dy_i dy_j
\end{aligned}
\tag{7.31}
$$

The current inner integral according to x_i is calculated as:

$$
\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_i\to f_{ij}}}x_i^2 + \frac{m_{\theta_i\to f_{ij}}}{\sigma^2_{x_i\to f_{ij}}}x_i\right\}\times \\
&\quad \exp\left\{-\frac{x_i^2 - 2x_jx_i - 2z_{ij}\lambda_{ij}x_i - 2c\lambda_{ij}\theta_i x_i}{2\sigma_d^2}\right\}\times \\
&\quad \exp\left\{-\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\lambda_{ij}\, x_i\, \theta_i\right\} dx_i \\
&\propto \exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)}\right]\theta_i^2\right. \\
&\qquad + \frac{c\,\lambda_{ij}\sigma_d^2 m_{x_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)}\,\theta_i \\
&\qquad \left. + \frac{c\,\lambda_{ij}\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)}\left(x_j + \lambda_{ij}z_{ij}\right)\,\theta_i\right\}
\end{aligned}
\tag{7.32}
$$

Using 7.32 in 7.31, the second becomes:

$$
\begin{aligned}
&\exp\left\{-\frac{c^2\theta_i^2 + 2z_{ij}\,c\,\theta_i}{2\sigma_d^2}\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right]\theta_i^2 + \frac{c^2 m_{\theta_j\to f_{ij}}}{\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}}\,\theta_i\right.\\
&\qquad\left. +\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}z_{ij}\,\theta_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)}\right]\theta_i^2\right.\\
&\left.+\frac{c\,\lambda_{ij}\sigma_d^2 m_{x_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)}\,\theta_i + \frac{c\,\lambda_{ij}\sigma^2_{x_i\to f_{ij}}z_{ij}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)}\lambda_{ij}\,\theta_i\right\}\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_ij}}y_j^2+\frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}\gamma_{ij}y_j+2c\gamma_{ij}\theta_i y_i}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\gamma_{ij}\,y_j\,\theta_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2+\frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\}\exp\left\{-2c\gamma_{ij}\theta_i y_i-\frac{y_i^2-2y_iy_j-2z_{ij}\gamma_{ij}y_i}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{-\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\gamma_{ij}\,y_i\,\theta_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_j\to f_{ij}}}x_j^2+\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}}x_j\right\}\exp\left\{-\frac{x_j^2+2z_{ij}c\lambda_{ij}x_j+2c\lambda_{ij}\theta_i x_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\lambda_{ij}\,x_j\,\theta_i\right\}\times\\
&\quad\exp\left\{\frac{c\,\lambda_{ij}\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)}\,x_j\,\theta_i\right\}dx_j dy_i dy_j
\end{aligned}
\tag{7.33}
$$

Now, the inner integral according to x_j is solved as:

$$
\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{x_j\to f_{ij}}}x_j^2+\frac{m_{x_j\to f_{ij}}}{\sigma^2_{x_j\to f_{ij}}}x_j\right\}\exp\left\{-\frac{x_j^2+2z_{ij}c\lambda_{ij}x_j+2c\lambda_{ij}\theta_i x_j}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\lambda_{ij}\,x_j\,\theta_i\right\}\times\\
&\quad\exp\left\{\frac{c\,\lambda_{ij}\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)}\,x_j\,\theta_i\right\}dx_j\\
&\propto \exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\sigma_d^4}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_j\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)}\right]\theta_i^2\right.\\
&\qquad-\frac{c\,\lambda_{ij}\sigma_d^4 m_{x_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)}\theta_i\\
&\qquad\left.+\frac{c\,\lambda_{ij}^2\sigma_d^2\sigma^2_{x_j\to f_{ij}}z_{ij}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_i\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{x_j\to f_{ij}}\right)}\theta_i\right\}
\end{aligned}
\tag{7.34}
$$

Putting 7.34 into 7.33, we get:

$$
\begin{aligned}
&\exp\left\{-\frac{c^2\theta_i^2 + 2z_{ij}\,c\,\theta_i}{2\sigma_d^2}\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\right]\theta_i^2 + \frac{c^2 m_{\theta_j\to f_{ij}}}{\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}}\,\theta_i\right.\\
&\qquad\left. + \frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)} z_{ij}\,\theta_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)}\right]\theta_i^2\right.\\
&\qquad + \frac{c\,\lambda_{ij}\sigma_d^2 m_{x_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)}\,\theta_i\\
&\qquad\left. + \frac{c\,\lambda_{ij}\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)}\lambda_{ij}\,z_{ij}\,\theta_i\right\}\times\\
&\exp\left\{-\frac{1}{2}\left[-\frac{c^2\lambda_{ij}^2\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_j\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_j\to f_{ij}}\right)}\right]\theta_i^2\right.\\
&\qquad - \frac{c\,\lambda_{ij}\sigma_d^4 m_{x_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_j\to f_{ij}}\right)}\,\theta_i\\
&\qquad\left. + \frac{c\,\lambda_{ij}^2\sigma_d^2\sigma^2_{x_j\to f_{ij}} z_{ij}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_j\to f_{ij}}\right)}\,\theta_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_ij}}y_j^2 + \frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2 + 2z_{ij}\gamma_{ij}y_j + 2c\gamma_{ij}\theta_i y_i}{2\sigma_d^2}\right\}\\
&\quad\exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}\gamma_{ij}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,y_j\,\theta_i\right\}\times\\
&\int\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2 + \frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\}\exp\left\{-2c\gamma_{ij}\theta_i y_i - \frac{y_i^2 - 2y_iy_j - 2z_{ij}\gamma_{ij}y_i}{2\sigma_d^2}\right\}\times\\
&\quad\exp\left\{-\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\,\gamma_{ij}\,y_i\,\theta_i\right\}dy_idy_j
\end{aligned}
\tag{7.35}
$$

The last two integrals according to y_i and y_j are absolutely symmetrical with the ones according to x_i and x_j respectively, that we just presented above, so their solutions are:

$$\begin{aligned}
&\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_i\to f_{ij}}}y_i^2 + \frac{m_{y_i\to f_{ij}}}{\sigma^2_{y_i\to f_{ij}}}y_i\right\} \\
&\quad \exp\left\{-\frac{y_i^2 - 2y_jy_i - 2z_{ij}\gamma_{ij}y_i - 2c\gamma_{ij}\theta_i y_i}{2\sigma_d^2}\right\}\times \\
&\quad \exp\left\{-\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\gamma_{ij}\, y_i\, \theta_i\right\} dy_i \\
&\propto \exp\left\{-\frac{1}{2}\left[-\frac{c^2\gamma_{ij}^2}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{y_i\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)}\right]\theta_i^2\right. \\
&\qquad + \frac{c\,\gamma_{ij}\sigma_d^2 m_{y_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)}\theta_i \\
&\qquad \left. + \frac{c\,\gamma_{ij}\sigma^2_{y_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)}\left(y_j + \gamma_{ij}z_{ij}\right)\theta_i\right\}
\end{aligned} \tag{7.36}$$

$$\int \exp\left\{-\frac{1}{2}\frac{1}{\sigma^2_{y_j\to f_{ij}}}y_j^2+\frac{m_{y_j\to f_{ij}}}{\sigma^2_{y_j\to f_{ij}}}y_j\right\}\exp\left\{-\frac{y_j^2+2z_{ij}c\gamma_{ij}y_j+2c\gamma_{ij}\theta_i y_j}{2\sigma_d^2}\right\}\times$$
$$\exp\left\{\frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)}\gamma_{ij}\,y_j\,\theta_i\right\}\times$$
$$\exp\left\{\frac{c\,\gamma_{ij}\sigma^2_{y_i\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)}y_j\,\theta_i\right\}dy_j$$
$$\propto \exp\left\{-\frac{1}{2}\left[-\frac{c^2\gamma_{ij}^2\sigma_d^4}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{y_j\to f_{ij}}}{\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}\right]\theta_i^2\right.$$
$$-\frac{c\,\gamma_{ij}\sigma_d^4 m_{y_j\to f_{ij}}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}\theta_i$$
$$\left.+\frac{c\,\gamma_{ij}^2\sigma_d^2\sigma^2_{y_j\to f_{ij}}z_{ij}}{\left(\sigma_d^2+c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_i\to f_{ij}}\right)\left(\sigma_d^2+\sigma^2_{y_j\to f_{ij}}\right)}\theta_i\right\} \tag{7.37}$$

So, utilizing the last two integrals of 7.36 and 7.37, the final expression for the message $\mu^{(l)}_{\theta_i \to f_{ij}}$ is:

$$
\begin{aligned}
&\mu^{(l)}_{\theta_i \to f_{ij}}(\theta_i) = \\
&\exp\Bigg\{ -\frac{1}{2}\Bigg[\frac{c^2}{\sigma_d^2} - \frac{c^2}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)} - \frac{c^2\lambda_{ij}^2}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_i\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\gamma_{ij}^2}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{y_i\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\lambda_{ij}^2\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{x_j\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{x_j\to f_{ij}}\right)} \\
&\qquad - \frac{c^2\gamma_{ij}^2\sigma_d^4}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)^2\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)^2}\frac{\sigma_d^2\sigma^2_{y_j\to f_{ij}}}{\left(\sigma_d^2 + \sigma^2_{y_j\to f_{ij}}\right)}\Bigg]\theta_i^2 \\
&\quad + \Bigg[-\frac{cz_{ij}}{\sigma_d^2} + \frac{c^2 m_{\theta_j\to f_{ij}}}{\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}} + \frac{c}{\sigma_d^2}\frac{c^2\sigma^2_{\theta_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)}z_{ij} \\
&\qquad + \frac{c\lambda_{ij}\sigma_d^2 m_{x_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)} + \frac{c\lambda_{ij}^2\sigma^2_{x_i\to f_{ij}}z_{ij}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)} \\
&\qquad - \frac{c\lambda_{ij}\sigma_d^4 m_{x_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_j\to f_{ij}}\right)} \\
&\qquad + \frac{c\lambda_{ij}^2\sigma_d^2\sigma^2_{x_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_i\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{x_j\to f_{ij}}\right)}z_{ij} \\
&\qquad + \frac{c\gamma_{ij}\sigma_d^2 m_{y_i\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)} + \frac{c\gamma_{ij}^2\sigma^2_{y_i\to f_{ij}}z_{ij}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)} \\
&\qquad - \frac{c\gamma_{ij}\sigma_d^4 m_{y_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_j\to f_{ij}}\right)} \\
&\qquad + \frac{c\gamma_{ij}^2\sigma_d^2\sigma^2_{y_j\to f_{ij}}}{\left(\sigma_d^2 + c^2\sigma^2_{\theta_j\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_i\to f_{ij}}\right)\left(\sigma_d^2 + \sigma^2_{y_j\to f_{ij}}\right)}z_{ij}\Bigg]\theta_i\Bigg\}
\end{aligned}
\tag{7.38}
$$

To obtain the same messages if node $j \in \mathcal{A}$, i.e., is an anchor, we simply make the following replacements

- $m_{x_i \to f_{ij}} = m_{x_i}$
- $m_{y_i \to f_{ij}} = m_{y_i}$
- $m_{\theta_i \to f_{ij}} = 0$
- $\sigma^2_{x_i \to f_{ij}} = 0$
- $\sigma^2_{y_i \to f_{ij}} = 0$
- $\sigma^2_{\theta_i \to f_{ij}} = 0$

and with some simple manipulations, we get the final expressions presented in the chapter of Joint Non-cooperative Estimation.

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