Observer-Based Output-Feedback Stabilization of Continua of Linear Hyperbolic PDEs Using Backstepping ⋆

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Abstract: We develop a boundary observer design for a class of continua of linear hyperbolic PDE systems, which are viewed as the continuum version of $n + m$ general heterodirectional hyperbolic systems as $n \to \infty$. The design relies on the introduction of a novel, continuum PDE backstepping transformation, which enables the construction of a Lyapunov functional for the estimation error system. We then introduce the respective non-collocated, output-feedback design employing the stabilizing continuum control kernels from Humaloja and Bekiaris-Liberis [2024]. (The observer-based output-feedback stabilization problem for the class of $\infty + 1$ hyperbolic systems can be solved as special case of the design procedure presented here.) Stability under the observer-based output-feedback law is established by using the Lyapunov functional construction for the estimation error system and proving well-posedness of the complete closed-loop system, which allows utilization of the separation principle. We illustrate the output-feedback design in simulation, via a numerical example for which the control and observer kernels can be computed in closed form.

Keywords: Backstepping observer, hyperbolic PDEs, PDE continua, output feedback.

1. INTRODUCTION

Continua of hyperbolic PDE systems can be viewed as continuum versions of certain, large-scale hyperbolic systems, featuring a large number of PDE state components (Humaloja and Bekiaris-Liberis [2024], Humaloja and Bekiaris-Liberis [2024]). A specific, theoretically and practically significant case of the latter, is the class of large-scale, $n+m$, heterodirectional, linear hyperbolic systems, which may be utilized to describe, for example, the dynamics of blood (Bikia [2021], Reymond et al. [2009]), traffic (Zhang et al. [2022], Herty and Klar [2003], Yu and Krstic [2021]), and water (Bastin and Coron [2016], Diagne et al. [2017]) flow networks, as well as the dynamics of epidemics transport (Guan et al. [2020], Kitsos et al. [2022]). In fact, certain control designs developed for stabilization of continua of hyperbolic PDE systems can be utilized for stabilization of the corresponding large-scale system when the number of state components is sufficiently large (Humaloja and Bekiaris-Liberis [2024], Humaloja and Bekiaris-Liberis [2024]). This is an important feature as it allows construction of stabilizing control kernels whose computational complexity does not grow with the number of state components (Humaloja and Bekiaris-Liberis [2025b,c]). A natural next step is to introduce a dual approach in which one constructs observer kernels for continua of hyperbolic systems, which could, in principle, be utilized as (approximate) observer kernels for the large-scale system counterpart (whose computational complexity remains tractable as the number of state components increases). Motivated by this and the fact that an observer-based output-feedback design is unavailable, in the present paper we address the problems of observer and output-feedback designs for such a class of continua of hyperbolic systems.

Full-state feedback laws for a class of continua of linear hyperbolic PDEs have been recently developed in Alleaume and Krstic [2025] and Humaloja and Bekiaris-Liberis [2024]. In particular, Alleaume and Krstic [2025] first addressed the problem of stabilization of a continuum version of the $n+1$ systems considered in Di Meglio et al. [2013] as $n \to \infty$, whereas in Humaloja and Bekiaris-Liberis [2024] we developed a feedback control design approach for stabilization of the continuum counterpart (as $n \to \infty$) of the $n+m$ hyperbolic systems considered in Hu et al. [2016]. The fact that the control design procedure developed for the continuum system in Alleaume and Krstic [2025] can be employed for stabilization of the large-scale $n + 1$ (for finite n) system, as it may provide stabilizing control kernels that can approximate to arbitrary accuracy the exact (constructed directly for the large-scale $n + 1$ system) backstepping kernels (for sufficiently large n), was established in Humaloja and Bekiaris-Liberis [2024, 2025c]. There exists no result heretofore addressing the problems of observer and output-feedback designs for such classes of continua of hyperbolic PDE systems.

In the present paper we develop a backstepping-based observer design methodology for a class of continua systems,

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which are viewed as the continuum version of $n+m$, linear hyperbolic systems as $n \to \infty$. Specifically, we address the dual to the control design problem from Humaloja and Bekiaris-Liberis [2024] in which we consider availability of m measurements, anti-collocated to the boundary where control is applied. Introducing a suitable target system we derive the continuum kernel equations, which are shown to be well-posed by recasting them in the form of the control kernel equations from Humaloja and Bekiaris-Liberis [2024]. Our choice of the target system enables construction of a Lyapunov functional for the estimation error system. We then introduce the respective observer-based output-feedback design combining the observer design developed here with the control design developed in Humaloja and Bekiaris-Liberis [2024]. The key in establishing stability of the complete closed-loop system is to show its well-posedness, which in turn allows us to employ the separation principle. The well-posedness proof relies on recasting the complete closed-loop system as an abstract system in output-feedback form and deriving the respective transfer function matrix. We provide consistent simulation results presenting a numerical example for which the control and observer kernels can be computed explicitly.

Notation. We use the standard notation $L^2(\Omega;\mathbb{R})$ for real-valued Lebesgue integrable functions on a domain $\Omega \in \mathbb{R}^d$ for some $d \geq 1$, and on one dimensional domains H^1 denotes the corresponding Sobolev space. We denote $E_c = L^2([0,1];L^2([0,1];\mathbb{R})) \times L^2([0,1];\mathbb{R}^m)$ and equip it with the inner product

$$\langle \left(\begin{smallmatrix} u_1 \\ \mathbf{v}_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} u_2 \\ \mathbf{v}_2 \end{smallmatrix}\right) \rangle_{E_c} = \int_0^1 \left(\int_0^1 u_1(x,y)u_2(x,y)dy + \sum_{j=1}^m v_1^j(x)v_2^j(x) \right) dx. \quad (1)$$

Moreover, we say that a system is exponentially stable on E_c if, for any initial condition $z_0 \in E_c$, the (weak) solution $z \in C([0,\infty);E_c)$ of the system satisfies $\|z(t)\|_{E_c} \leq Me^{-ct}\|z_0\|_{E_c}$ for some constants $M, c > 0$ that are independent of z_0. Finally, we denote by $\mathcal{T}$ the triangular set

$$\mathcal{T} = \left\{(x,\xi) \in [0,1]^2 : 0 \leq \xi \leq x \leq 1\right\}. \quad (2)$$

2. CONTINUA SYSTEMS OF HYPERBOLIC PDES

The considered class of continuum systems is of the form

$$u_t(t,x,y) + \lambda(x,y)u_x(t,x,y) = \int_0^1 \sigma(x,y,\eta)u(t,x,\eta)d\eta + \mathbf{W}(x,y)\mathbf{v}(t,x), \quad (3a)$$

$$\mathbf{v}_t(t,x) - \mathbf{\Lambda}(x)\mathbf{v}_x(t,x) = \int_0^1 \mathbf{\Theta}(x,y)u(t,x,y)dy + \mathbf{\Psi}(x)\mathbf{v}(t,x), \quad (3b)$$

with boundary conditions

$$u(t,0,y) = \mathbf{Q}(y)\mathbf{v}(t,0), \quad (4a)$$

$$\mathbf{v}(t,1) = \int_0^1 \mathbf{R}(y)u(t,1,y)dy + \mathbf{U}(t), \quad (4b)$$

and output $\mathbf{Y}(t) = \mathbf{v}(t,0)$, for almost every $y \in [0,1]$. Here we employ the matrix notation for $\mathbf{v}, \mathbf{U}, \mathbf{Y}, \mathbf{\Lambda}, \mathbf{\Theta}, \mathbf{\Psi}, \mathbf{W}, \mathbf{Q}$, and $\mathbf{R}$ for the sake of conciseness, that is, $\mathbf{v} = \left(v^j\right)_{j=1}^m$, $\mathbf{U} = \left(U^j\right)_{j=1}^m$, $\mathbf{Y} = \left(Y^j\right)_{j=1}^m$, and the parameters are as follows.

Assumption 1. The parameters of (3), (4) are such that

$$\mathbf{\Lambda} = \mathrm{diag}(\mu_j)_{j=1}^m \in C^1([0,1];\mathbb{R}^{m\times m}), \quad (5a)$$

$$\mathbf{\Theta} = (\Theta_j)_{j=1}^m \in C([0,1];L^2([0,1];\mathbb{R}^m)), \quad (5b)$$

$$\mathbf{\Psi} = (\Psi_{i,j})_{i,j=1}^m \in C([0,1];\mathbb{R}^{m\times m}), \quad (5c)$$

$$\mathbf{W} = [W_1 \ \cdots \ W_m] \in C([0,1];L^2([0,1];\mathbb{R}^{1\times m})), \quad (5d)$$

$$\mathbf{Q} = [Q_1 \ \cdots \ Q_m] \in L^2([0,1];\mathbb{R}^{1\times m}), \quad (5e)$$

$$\mathbf{R} = (R_j)_{j=1}^m \in L^2([0,1];\mathbb{R}^m), \quad (5f)$$

with $\lambda \in C^1([0,1]^2;\mathbb{R})$ and $\sigma \in C([0,1];L^2([0,1]^2;\mathbb{R}))$. Moreover, $\mu_m(x) > 0$ and $\lambda(x,y) > 0$ uniformly for all $x, y \in [0,1]$, and additionally

$$\min_{x\in[0,1]} \mu_j(x) > \max_{x\in[0,1]} \mu_{j+1}(x), \quad (6)$$

for all $j = 1, \ldots, m-1$. Finally, $\psi_{j,j} = 0$ for all $j = 1, \ldots, m$. [1]

3. OUTPUT-FEEDBACK STABILIZATION

3.1 Control Law and Observer Design

The control law to stabilize (3), (4) is of the form

$$\mathbf{U}(t) = \int_0^1 \int_0^1 \mathbf{K}(1,\xi,y)\hat{u}(t,\xi,y)dyd\xi + \int_0^1 \mathbf{L}(1,\xi)\hat{\mathbf{v}}(t,\xi)d\xi - \int_0^1 \mathbf{R}(y)\hat{u}(t,1,y)dy, \quad (7)$$

where $\mathbf{K}, \mathbf{L}$ are the backstepping control kernels (see Humaloja and Bekiaris-Liberis [2024]) and $\hat{u}, \hat{\mathbf{v}}$ is the observer state satisfying the dynamics

$$\hat{u}_t(t,x,y) + \lambda(x,y)\hat{u}_x(t,x,y) + \mathbf{P}_+(x,y)\tilde{\mathbf{v}}(t,0) = \int_0^1 \sigma(x,y,\eta)\hat{u}(t,x,\eta)d\eta + \mathbf{W}(x,y)\hat{\mathbf{v}}(t,x), \quad (8a)$$

$$\hat{\mathbf{v}}_t(t,x) - \mathbf{\Lambda}(x)\hat{\mathbf{v}}_x(t,x) + \mathbf{P}_-(x)\tilde{\mathbf{v}}(t,0) = \int_0^1 \mathbf{\Theta}(x,y)\hat{u}(t,x,y)dy + \mathbf{\Psi}(x)\hat{\mathbf{v}}(t,x), \quad (8b)$$

with boundary conditions

$$\hat{u}(t,0,y) = \mathbf{Q}(y)\mathbf{v}(t,0), \quad (9a)$$

$$\hat{\mathbf{v}}(t,1) = \int_0^1 \mathbf{R}(y)\hat{u}(t,1,y)dy + \mathbf{U}(t), \quad (9b)$$

where we denote $\tilde{\mathbf{v}} = \hat{\mathbf{v}} - \mathbf{v}$ and $\mathbf{P}_+, \mathbf{P}_-$ are given by

$$\mathbf{P}_+(x,y) = \mathbf{M}(x,0,y)\mathbf{\Lambda}(0), \quad (10a)$$

$$\mathbf{P}_-(x) = \mathbf{N}(x,0)\mathbf{\Lambda}(0), \quad (10b)$$

[1] This comes without loss of generality, as such terms can be removed using a change of variables (see, e.g., Hu et al. [2016, 2019]).

where $\mathbf{M} \in L^\infty(\mathcal{T}; L^2([0,1]; \mathbb{R}^{1\times m})), \mathbf{N} \in L^\infty(\mathcal{T}; \mathbb{R}^{m\times m}))$, is the solution to the observer kernel equations

$$\lambda(x,y)\mathbf{M}_x(x,\xi,y) - \mathbf{M}_\xi(x,\xi,y)\mathbf{\Lambda}(\xi) - \mathbf{M}(x,\xi,y)\mathbf{\Lambda}'(\xi) = \int_0^1 \sigma(\xi,y,\eta)\mathbf{M}(x,\xi,\eta)d\eta + \mathbf{W}(\xi,y)\mathbf{N}(x,\xi), \tag{11a}$$

$$\mathbf{\Lambda}(x)\mathbf{N}_x(x,\xi) + \mathbf{N}_\xi(x,\xi)\mathbf{\Lambda}(\xi) + \mathbf{N}(x,\xi)\mathbf{\Lambda}'(\xi) = \int_0^1 \mathbf{\Theta}(\xi,y)\mathbf{M}(x,\xi,y)dy + \mathbf{\Psi}(\xi)\mathbf{N}(x,\xi), \tag{11b}$$

with boundary conditions

$$\mathbf{W}(x,y) = \mathbf{M}(x,x,y)\mathbf{\Lambda}(x) + \lambda(x,y)\mathbf{M}(x,x,y), \tag{12a}$$

$$\mathbf{\Psi}(x) = \mathbf{N}(x,x)\mathbf{\Lambda}(x) - \mathbf{\Lambda}(x)\mathbf{N}(x,x), \tag{12b}$$

$$N_{i,j}(1,\xi) = \int_0^1 R_i(y)M_j(1,\xi,y)dy, \quad \forall i \geq j, \tag{12c}$$

$$N_{i,j}(x,0) = n_{i,j}(x), \quad \forall i < j, \tag{12d}$$

where $n_{i,j}(x)$ are arbitrary due to (12d) being an artificial boundary condition to guarantee well-posedness of the observer kernel equations; similarly to the $n+m$ case in Hu et al. [2016]. We choose $n_{i,j}(x)$ such that

$$n_{i,j}(0) = \frac{\Psi_{i,j}(0)}{\mu_j(0) - \mu_i(0)}, \tag{13}$$

in order to make the artificial boundary condition compatible with (12b) at $(0,0)$. Note that the boundary conditions for $N(1,1)$ are, in general, overdetermined due to (12b) and (12c), (12a), which stems potential discontinuities in the $\mathbf{N}$ kernels.

3.2 Well-Posedness of Observer Kernel Equations

Theorem 1. Under Assumption 1, the observer kernel equations (11), (12) have a well-posed solution $\mathbf{M} \in L^\infty(\mathcal{T}; L^2([0,1]; \mathbb{R}^{1\times m})), \mathbf{N} \in L^\infty(\mathcal{T}; \mathbb{R}^{m\times m}))$. Moreover, the solution is piecewise continuous in $(x,\xi) \in \mathcal{T}$, where the set of discontinuities is of measure zero.

Proof. The proof is based on transforming the kernel equations (11)–(13) into an analogous form with the respective control kernel equations, which have been shown to be well-posed in [Humaloja and Bekiaris-Liberis 2024, Thm 2]. This transformation is achieved by introducing alternative variables

$$\bar{\mathbf{M}}(\chi,\zeta,y) = \mathbf{M}(1-\zeta, 1-\chi, y) = \mathbf{M}(x,\xi,y), \tag{14a}$$

$$\bar{\mathbf{N}}(\chi,\zeta) = \mathbf{N}(1-\zeta, 1-\chi) = \mathbf{N}(x,\xi), \tag{14b}$$

and analogously introducing the alternative parameters $\bar{\mu}, \bar{\lambda}, \bar{\sigma}, \bar{\Theta}, \bar{W}, \bar{\Psi}, \bar{n}_{i,j}$ based on the coordinate transform $(x,\xi) \to (1-\zeta, 1-\chi)$, so that (χ,ζ) is the mirror image of (x,ξ) with respect to the line $x+\xi = 1$, and hence, $(x,\xi) \in \mathcal{T}$ is mirrored into $(\chi,\zeta) \in \mathcal{T}$. Inserting these new coordinates into the observer kernel equations (11), (12), we obtain new kernel equations analogous to the control kernel equations in [Humaloja and Bekiaris-Liberis 2024, (27)–(33)], and hence, the claim follows by [Humaloja and Bekiaris-Liberis 2024, Thm 2]. For more details, we refer the reader to the proof of [Humaloja and Bekiaris-Liberis 2025a, Thm 1]. □

3.3 Stability of the Closed-Loop System

Theorem 2. Under Assumption 1, the closed-loop system (3), (4) under the observer-based output-feedback control law (7)–(9) is exponentially stable on $E_c \times E_c$.

The proof is presented at the end of this subsection by utilizing the well-posedness of the closed-loop system given by Proposition 1 in Appendix A, which allows utilization of the separation principle. We first prove the exponential stability of the estimation error dynamics.

Lemma 1. Under Assumption 1, the estimation error dynamics for $\tilde{u} = \hat{u} - u$ and $\tilde{\mathbf{v}}$ given by

$$\tilde{u}_t(t,x,y) + \lambda(x,y)\tilde{u}_x(t,x,y) + \mathbf{P}_+(x,y)\tilde{\mathbf{v}}(t,0) = \int_0^1 \sigma(x,y,\eta)\tilde{u}(t,x,\eta)d\eta + \mathbf{W}(x,y)\tilde{\mathbf{v}}(t,x), \tag{15a}$$

$$\tilde{\mathbf{v}}_t(t,x) - \mathbf{\Lambda}(x)\tilde{\mathbf{v}}_x(t,x) + \mathbf{P}_-(x)\tilde{\mathbf{v}}(t,0) = \int_0^1 \mathbf{\Theta}(x,y)\tilde{u}(t,x,y)dy + \mathbf{\Psi}(x)\tilde{\mathbf{v}}(t,x), \tag{15b}$$

with boundary conditions

$$\tilde{u}(t,0,y) = 0, \tag{16a}$$

$$\tilde{\mathbf{v}}(t,1) = \int_0^1 \mathbf{R}(y)\tilde{u}(t,1,y)dy. \tag{16b}$$

are exponentially stable on E_c.

Proof. We transform (15), (16) into the following target system

$$\tilde{\alpha}_t(t,x,y) + \lambda(x,y)\tilde{\alpha}_x(t,x,y) = \int_0^1 \sigma(x,y,\eta)\tilde{\alpha}(t,x,\eta)d\eta + \int_0^1\int_0^x D_+(x,\xi,y,\eta)\tilde{\alpha}(t,\xi,\eta)d\xi d\eta, \tag{17a}$$

$$\tilde{\boldsymbol{\beta}}_t(t,x) - \mathbf{\Lambda}(x)\tilde{\boldsymbol{\beta}}_x(t,x) = \int_0^1 \mathbf{\Theta}(x,y)\tilde{\alpha}(t,x,y)dy + \int_0^1\int_0^x \mathbf{D}_-(x,\xi,y)\tilde{\alpha}(t,\xi,y)d\xi dy, \tag{17b}$$

with boundary conditions

$$\tilde{\alpha}(t,0,y) = 0, \tag{18a}$$

$$\tilde{\boldsymbol{\beta}}(t,1) = \int_0^1 \mathbf{R}(y)\tilde{\alpha}(t,1,y)dy - \int_0^1 \mathbf{H}(\xi)\tilde{\boldsymbol{\beta}}(t,\xi)d\xi, \tag{18b}$$

where $D_+ \in L^\infty(\mathcal{T}; L^2([0,1]^2; \mathbb{R})), \mathbf{D}_- \in L^\infty(\mathcal{T}; L^2([0,1]; \mathbb{R}^m))$, and $\mathbf{H} \in L^\infty([0,1]; \mathbb{R}^{m\times m})$ is strictly upper triangular, i.e., $H_{i,j} = 0$ for all $i \geq j$. The transformation is of the form

$$\tilde{u}(t,x,y) = \tilde{\alpha}(t,x,y) + \int_0^x \mathbf{M}(x,\xi,y)\tilde{\boldsymbol{\beta}}(t,\xi)d\xi, \tag{19a}$$

$$\tilde{\mathbf{v}}(t,x) = \tilde{\boldsymbol{\beta}}(t,x) + \int_0^x \mathbf{N}(x,\xi)\tilde{\boldsymbol{\beta}}(t,\xi)d\xi, \tag{19b}$$

where $\mathbf{M}, \mathbf{N}$ are the observer kernels. For the derivation of the observer kernel equations and the expressions for $D_+, \mathbf{D}_+$, and $\mathbf{H}$, we refer to [Humaloja and Bekiaris-Liberis 2025a, App. A].

We then establish that the transform (19) is boundedly invertible, which follows as (19b) is a Volterra equation of second kind for $\tilde{\boldsymbol{\beta}}(\cdot, x)$ in terms of $\tilde{\mathbf{v}}(\cdot, x)$ and $\mathbf{N}$. Thus, by [Hochstadt 1989, Thm 2.3.6], (19b) has a unique solution $\tilde{\boldsymbol{\beta}}(t, \cdot) \in L^2([0,1]; \mathbb{R}^m)$ for all $t \geq 0$. Thereafter, $\tilde{\alpha}(\cdot, t) \in L^2([0,1]; L^2([0,1]; \mathbb{R}))$ is uniquely determined by (19a) for all $t \geq 0$. Thus, the transform (19) is boundedly invertible, and hence, the exponential stability of (15), (16) is equivalent to the exponential stability of (17), (18).

In order to show that (17), (18) is exponentially stable, consider a Lyapunov functional with parameters $\delta, \mathbf{B} = \mathrm{diag}(B_1, \ldots, B_m) > 0$ of the form

$$\begin{aligned} V(t) &= \int_0^1 \int_0^1 e^{-\delta x} \frac{\tilde{\alpha}^2(t,x,y)}{\lambda(x,y)} dy dx \\ &\quad + \int_0^1 e^{\delta x} \tilde{\boldsymbol{\beta}}^T(t,x) \mathbf{B} \boldsymbol{\Lambda}^{-1}(x) \tilde{\boldsymbol{\beta}}(t,x) dx. \end{aligned} \tag{20}$$

After computing $\dot{V}(t)$ and integrating by parts in x (see [Humaloja and Bekiaris-Liberis 2025a, (31)–(42)] for details), we obtain that $\dot{V}(t)$ can be made negative definite by choosing any sufficiently large δ and, subsequently, appropriate weights $(B_j)_{j=1}^m$. Hence, the target system (17), (18) is exponentially stable, which by the invertibility of the transform (19) implies the exponential stability of (15), (16). □

Proof of Theorem 2. Since the closed-loop system (3), (4), (7)–(9), is well-posed by Proposition 1, we can introduce a change of variables $\tilde{z} = \hat{z} - z$ and write the closed-loop system equivalently, utilizing the notation of Appendix A, as

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\tilde{z}}(t) \end{bmatrix} = \begin{bmatrix} A_{-1} + B_Q \mathcal{C} + B\mathcal{K} & -B\mathcal{C}_R + B\mathcal{K} \\ 0 & A_{-1} + B\mathcal{C}_R + P\mathcal{C} \end{bmatrix} \begin{bmatrix} z(t) \\ \tilde{z}(t) \end{bmatrix}, \tag{21}$$

where $\dot{z}(t) = (A_{-1} z(t) + B_Q \mathcal{C} + B\mathcal{K})\, z(t)$ corresponds to the closed-loop dynamics of (3), (4) under the backstepping state feedback law $\mathbf{U}(t) = \mathcal{K} z(t) - \mathcal{C}_R z(t)$ and $\dot{\tilde{z}}(t) = (A_{-1} + B\mathcal{C}_R + P\mathcal{C})\, \tilde{z}(t)$ corresponds to the estimation error dynamics (15), (16). As these dynamics are exponentially stable by [Humaloja and Bekiaris-Liberis 2024, Thm 1] and Lemma 1, respectively, the exponential stability of the closed-loop system follows, e.g., by the Gearhart—Prüss—Greiner Theorem [Engel and Nagel 2000, Thm V.1.11] and the decay rate is determined by the smaller one of the diagonal dynamics.

4. NUMERICAL EXAMPLE

Consider the following parameters for $x, y, \eta \in [0,1]$

$$\lambda(x,y) = 1, \quad \mu_1(x) = 2, \quad \mu_2(x) = 1, \tag{22a}$$

$$\sigma(x,y,\eta) = x^3(x+1)\left(y - \frac{1}{2}\right)\eta(\eta - 1), \tag{22b}$$

$$W_1(x,y) = 3\left(y - \frac{1}{2}\right), \quad W_2(x,y) = 2\left(y - \frac{1}{2}\right), \tag{22c}$$

$$\theta_1(x,y) = -3y(y-1), \quad \theta_2(x,y) = -2y(y-1), \tag{22d}$$

$$\psi_{i,j}(x) = 0, \quad i,j \in \{1,2\}, \tag{22e}$$

$$Q_1(y) = 8\left(y - \frac{1}{2}\right), \quad Q_2(y) = -8(y-2), \tag{22f}$$

$$R_1(y) = \cos(2\pi y), \quad R_2(y) = 2y(y+5). \tag{22g}$$

The observer kernel equations (11), (12) have explicit solution

$$M_1^1(x,\xi,y) = \left(y - \frac{1}{2}\right), \tag{23a}$$

$$M_1^2(x,\xi,y) = e^{x - \frac{1}{2}\xi - 1}\left(y - \frac{1}{2}\right), \tag{23b}$$

$$M_2^2(x,\xi,y) = e^{x-\xi}\left(y - \frac{1}{2}\right), \tag{23c}$$

$$N_{1,1}^1(x,\xi) = 0, \quad N_{1,1}^2(x,\xi) = 0, \tag{23d}$$

$$N_{2,1}^1(x,\xi) = 0, \quad N_{2,1}^2(x,\xi) = e^{x - \frac{1}{2}\xi - 1}, \tag{23e}$$

$$N_{1,2}^2(x,\xi) = 0, \quad N_{2,2}^2(x,\xi) = e^{x-\xi}, \tag{23f}$$

where $M_1^\star(\cdot, y), N_{1,1}^\star$, and $N_{2,1}^\star$ are defined on $\mathcal{T}_1^1 = \{(x,\xi) \in \mathcal{T} : \xi \geq 2x - 1\}$ and $\mathcal{T}_1^2 = \{(x,\xi) \in \mathcal{T} : \xi \leq 2x - 1\}$ for the respective superindex $\star = 1, 2$, while $M_2^2(\cdot, y), N_{2,1}^2$ and $N_{2,2}^2$ are defined on $\mathcal{T}_2^2 = \mathcal{T}$, for each $y \in [0,1]$. Note the discontinuity in $N_{2,1}$ along $\xi = 2x - 1$. The control kernels $\mathbf{K}, \mathbf{L}$ are the same as in the example considered in [Humaloja and Bekiaris-Liberis 2024, Sect. 6] (despite the parameters being slightly different).

For the simulation, the continuum system (3), (4) and the observer (8), (9) are approximated by a grid of 50 points in y and 128 points in x, where we use finite differences to approximate the differential operators. The resulting ODE approximating the closed-loop system (3), (4), (7)–(9) is solved using `ode45` in MATLAB. The initial conditions are $u_0(x,y) = \sin(\pi x), v_0^1(x) = v_0^2(x) = \sin(\pi x)$ and $\hat{u}_0 = 0, \hat{v}_0^1 = \hat{v}_0^2 = 0$.

The simulation results are shown in Figs. 1 and 2 for $t \in [0, 6]$. Fig. 1 shows the output estimation errors $\tilde{\mathbf{Y}}(t) = \tilde{\mathbf{v}}(t, 0)$ and controls (7), demonstrating that, after the initial transients, both $\mathbf{U}$ and $\tilde{\mathbf{Y}}$ converge to zero by approximately five seconds in the simulation. Fig. 2 shows the solution components $u(t,x,y)$ evaluated along $y = 1$ and $v^1(t,x)$, which also converge to zero by approximately five seconds in the simulation, where particularly $u(t,x,1)$ exhibits some large transients due to the unstable open-loop dynamics. Regardless, based on Figs. 1 and 2, the control law (7) stabilizes the open-loop unstable continuum system and the estimation error decays to zero as well.

5. CONCLUSIONS AND ONGOING WORK

We developed an observer design approach for a class of continua of linear hyperbolic PDE systems. The design relies on introduction of a continuum backstepping

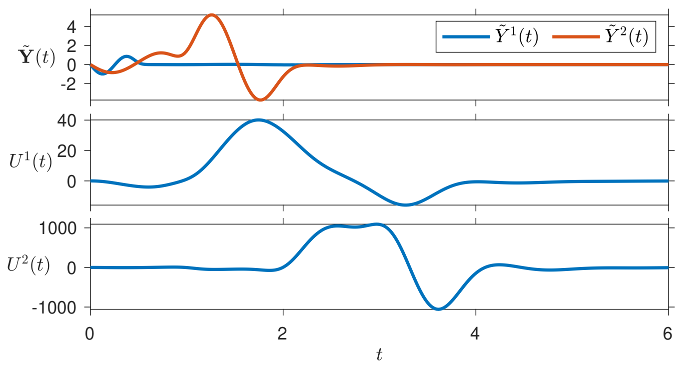

Fig. 1. The controls $\mathbf{U}(t)$ based on the control law (7) and the output estimation errors $\tilde{\mathbf{Y}}(t) = \tilde{\mathbf{v}}(t, 0)$.

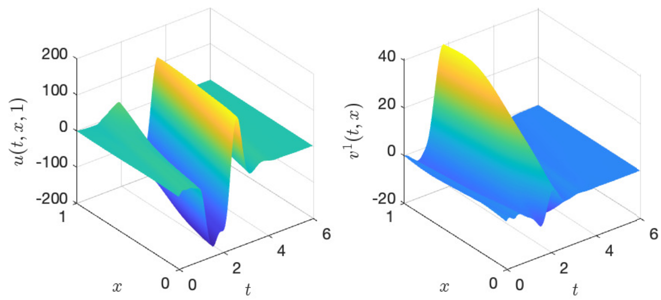

Fig. 2. The solution components $u(t, x, 1)$ and $v^1(t, x)$.

transformation and of a suitable target system for the estimation error system, which enables construction of a Lyapunov functional for the estimation error dynamics. We also introduced the respective non-collocated output-feedback design, combining the observer design developed here with the control design from Humaloja and Bekiaris-Liberis [2024]. Stability of the closed-loop system under the proposed output-feedback law was proven by showing well-posedness of the complete closed-loop system, which allows us to employ the separation principle. A numerical example was devised to illustrate the output-feedback design in simulation, in which, in particular, the control and observer kernels are obtained in closed form.

We are currently investigating the collocated, observer-based output-feedback design problem, as well as the problem of utilization of the observer kernels constructed here in an observer design for the large-scale $n+m$ system counterpart, in order to obtain observer kernels whose computational complexity does not grow with the number of state components.

Appendix A. WELL-POSEDNESS OF (3), (4), (7)–(9)

Proposition 1. The closed-loop system (3), (4), (7)–(9) is well-posed on $E_c \times E_c$.

Proof. In order to write the closed-loop system (3), (4), (7)–(9) more compactly as an abstract Cauchy problem, define $z(t) = (u(t, \cdot, \cdot), \mathbf{v}(t, \cdot))$, and

$$Az(t) = \begin{bmatrix} -\lambda\partial_x & 0 \\ 0 & \mathbf{\Lambda}\partial_x \end{bmatrix} z(t) + Sz(t), \tag{A.1}$$

where $Sz(t)$ corresponds to the right-hand side of (3) and the domain of A is defined as

$$\mathcal{D}(A) = \{z \in H^1([0,1]; L^2([0,1];\mathbb{R}) \times \mathbb{R}^m) : u(0) = 0, \mathbf{v}(1) = 0\}. \tag{A.2}$$

Moreover, denote by A_{-1} the (unique) extension of A from E_c to the dual space of $\mathcal{D}(A)$ with respect to E_c, which exists by [Tucsnak and Weiss 2009, Prop. 2.10.2] due to $\mathcal{D}(A)$ being dense in E_c. In order to express the boundary couplings, define control operators B, B_Q according to the boundary traces in (4) as [Tucsnak and Weiss 2009, Rem. 10.1.6]

$$B = \begin{bmatrix} 0 \\ \mathbf{\Lambda}\delta_1 \end{bmatrix}, \qquad B_Q = \begin{bmatrix} \lambda\delta_0\mathbf{Q} \\ 0 \end{bmatrix}, \tag{A.3a}$$

where $\delta_\star$ denotes the Dirac delta function at $x = \star$, and output operators $\mathcal{C}, \mathcal{C}_R$ as

$$\mathcal{C}z(t) = \mathbf{v}(0,t), \qquad \mathcal{C}_R z(t) = \int_0^1 \mathbf{R}(y)u(t,1,y)dy. \tag{A.4}$$

Moreover, denote by $P, \mathcal{K}$ the backstepping observer and controller gains, so that the closed-loop system (3), (4), (7)–(9) can be written as an abstract Cauchy problem

$$\begin{aligned} \begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} &= \begin{bmatrix} A_{-1} + B\mathcal{C}_R + B_Q\mathcal{C} & -B\mathcal{C}_R + B\mathcal{K} \\ P\mathcal{C} + B_Q\mathcal{C} & A_{-1} - P\mathcal{C} + B\mathcal{K} \end{bmatrix} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} \\ &= \left(\begin{bmatrix} A_{-1} & 0 \\ 0 & A_{-1} \end{bmatrix} + \begin{bmatrix} B & B & B_Q & 0 \\ B & 0 & B_Q & -P \end{bmatrix} \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{C}_R & -\mathcal{C}_R \\ \mathcal{C} & 0 \\ -\mathcal{C} & \mathcal{C} \end{bmatrix} \right) \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} \\ &=: \left(A^{\mathrm{e}}_{-1} + B^{\mathrm{e}}\mathcal{C}^{\mathrm{e}}\right) \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}. \end{aligned} \tag{A.5}$$

Thus, (A.5) can be written in an output feedback form, and by [Jacob and Zwart 2012, Thm 13.1.12] the closed-loop system is well-posed if the inverse of $I - \mathbf{G}^{\mathrm{e}}(s)$ exists and is bounded for all $\mathrm{Re}(s)$ sufficiently large, where $\mathbf{G}^{\mathrm{e}}$ is the transfer function of the triple $(A^{\mathrm{e}}, B^{\mathrm{e}}, \mathcal{C}^{\mathrm{e}})$. By virtue of [Jacob and Zwart 2012, Lem. 13.1.14], the well-posedness of the closed-loop system is independent of the (bounded) in-domain coupling terms $Sz(t), S\hat{z}(t)$, meaning that it suffices to consider the transfer function $\bar{\mathbf{G}}^{\mathrm{e}}$, which can be computed by [Cheng and Morris 2003, Thm 2.9] from

$$su(s,x,y) = -\lambda(x,y)u_x(s,x,y), \tag{A.6a}$$
$$s\mathbf{v}(s,x) = \mathbf{\Lambda}(x)\mathbf{v}_x(s,x), \tag{A.6b}$$
$$s\hat{u}(s,x,y) = -\lambda(x,y)\hat{u}_x(s,x,y) - \mathbf{P}_+(x,y)\mathbf{U}_4(s), \tag{A.6c}$$
$$s\hat{\mathbf{v}}(s,x) = \mathbf{\Lambda}(x)\hat{\mathbf{v}}_x(s,x) - \mathbf{P}_-(x)\mathbf{U}_4(s), \tag{A.6d}$$
$$\mathbf{v}(s,1) = \mathbf{U}_1(s) + \mathbf{U}_2(s), \tag{A.6e}$$
$$\hat{\mathbf{v}}(s,1) = \mathbf{U}_1(s), \tag{A.6f}$$
$$u(s,0,y) = \mathbf{Q}(y)\mathbf{U}_3(s), \tag{A.6g}$$
$$\hat{u}(s,0,y) = \mathbf{Q}(y)\mathbf{U}_3(s), \tag{A.6h}$$

$$\mathbf{Y}_1(s) = \int_0^1 \int_0^1 \mathbf{K}(1,\xi,y)\hat{u}(s,\xi,y)dyd\xi + \int_0^1 \mathbf{L}(1,\xi)\hat{\mathbf{v}}(s,\xi)d\xi, \tag{A.6i}$$

$$\mathbf{Y}_2(s) = \int_0^1 \mathbf{R}(y)\left(u(s,1,y) - \hat{u}(s,1,y)\right)dy, \tag{A.6j}$$

$$\mathbf{Y}_3(s) = \mathbf{v}(s,0), \qquad \mathbf{Y}_4(s) = \hat{\mathbf{v}}(s,0) - \mathbf{v}(s,0), \tag{A.6k}$$

where $(\mathbf{Y}_i(s))_{i=1}^4 = \bar{\mathbf{G}}^{\mathrm{e}}(s)\,(\mathbf{U}_i(s))_{i=1}^4$. Since $P, \mathcal{K}$ are bounded operators, the components of $\bar{\mathbf{G}}^{\mathrm{e}}$ involving them necessarily tend to zero as $\mathrm{Re}(s) \to \infty$. The remaining components can be computed based on the general solution

$$u(s,x,y) = a(y)\exp\left(-s\int_0^x \lambda(\zeta,y)d\zeta\right), \tag{A.7a}$$

$$\mathbf{v}(s,x) = \exp\left(s\int_0^x \mathbf{\Lambda}(\zeta)d\zeta\right)\mathbf{b}, \tag{A.7b}$$

to (A.6a), (A.6b) (respectively for $\hat{u}, \hat{\mathbf{v}}$), where the coefficients $a, \mathbf{b}$ are solved for from the boundary conditions of (A.6). We obtain (for $\mathbf{U}_4 \equiv 0$)

$$\begin{bmatrix}\mathbf{Y}_2(s)\\ \mathbf{Y}_3(s)\\ \mathbf{Y}_4(s)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0\\ \exp\left(-s\int_0^1 \mathbf{\Lambda}(x)dx\right) & \exp\left(-s\int_0^1 \mathbf{\Lambda}(x)dx\right) & 0\\ 0 & -\exp\left(-s\int_0^1 \mathbf{\Lambda}(x)dx\right) & 0 \end{bmatrix}\begin{bmatrix}\mathbf{U}_1(s)\\ \mathbf{U}_2(s)\\ \mathbf{U}_3(s)\end{bmatrix}, \tag{A.8}$$

which concludes that $\mathrm{Re}\,\bar{\mathbf{G}}_{\mathrm{e}}(s) \to 0$ as $\mathrm{Re}(s) \to \infty$ due to the diagonal matrix $\mathbf{\Lambda}$ satisfying $\mathbf{\Lambda} > 0$ by Assumption 1. Thus, the inverse of $I - \bar{\mathbf{G}}^{\mathrm{e}}(s)$ exists and is bounded when $\mathrm{Re}(s)$ is sufficiently large, and hence, the closed-loop system is well-posed by [Jacob and Zwart 2012, Thm 13.1.12, Lem. 13.1.14]. That is, for all $z_0, \hat{z}_0 \in E_c$, there exists a unique solution $z, \hat{z} \in C([0,\infty); E_c)$ to (A.5). □

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