Stabilization of an unstable reaction-diffusion PDE with input delay despite state and input quantization

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***Abstract*— We solve the global asymptotic stability problem of an unstable reaction-diffusion Partial Differential Equation (PDE) subject to input delay and state quantization developing a switched predictor-feedback law. To deal with the input delay, we reformulate the problem as an actuated transport PDE coupled with the original reaction-diffusion PDE. Then, we design a quantized predictor-based feedback mechanism that employs a dynamic switching strategy to adjust the quantization range and error over time. The stability of the closed-loop system is proven properly combining backstepping with a small-gain approach and input-to-state stability techniques, for deriving estimates on solutions, despite the quantization effect and the system's instability. We also extend this result to the input quantization case.**

I. Introduction

The reaction-diffusion PDE is widely employed across various fields, including biology [2], chemistry [10], and epidemiology [11] to model phenomena such as heat distribution, chemical reactions, and the spread of diseases. Due to its broad applicability, the reaction-diffusion PDE has been extensively studied, in particular, in terms of stabilization under digital implementation effects, see, for example, [6], [14], [16], [21], [25], [26], [28] and references therein. In digital implementations of feedback laws for reaction-diffusion PDEs, the resulting closed-loop systems may be subject to input delay and quantization. Consider, for example, the case of epidemics spreading described by a reaction diffusion PDE system [11], which may be subject to delay, due to, e.g., testing or measures imposition lags [24], [32] and quantization, due to, e.g., discrete quarantine measures imposition (as implementation of measures chosen from a continuum are practically infeasible). The presence of input delay and quantization may deteriorate performance or even destabilize the closed-loop system.

For this reason, besides results on control of systems under quantization, see, for instance, [3], [4], [23] that examine linear systems with quantized measurements and [22] that addresses general nonlinear systems under quantization, there are results dealing with the effect of quantization in delay systems. In particular, [8] focuses on linear time-delay systems with saturation, [9] investigates logarithmic quantizers within an event-based control framework, and [5] considers nonlinear systems with state delay and quantization. Stabilization of various classes of PDEs under static quantization has also been addressed. In particular, [7] and [29] address first-order hyperbolic systems, whereas [16] and [27] focus on parabolic systems. Dynamic quantizers with adjustable "zoom" parameter have been designed for first-order hyperbolic systems in [1], for a linear ODE (Ordinary Differential Equation)-transport PDE cascade in [17], and for a general class of infinite-dimensional, discrete-time systems in [30]. One of the key differences between static and dynamic quantizers lies in that the latter can guarantee global asymptotic stability, which requires however assuming that the quantizer's range/error can be dynamically adjusted.

In this paper, we develop a switched predictor-feedback law for simultaneous compensation of input delay and state/input quantization. The feedback law is essentially a quantized version of the nominal backstepping/predictor-feedback law from [20], which is constructed by properly designing the adjustable parameter of the quantizer in a time-triggered manner, as it is performed in [17] for ODE systems with input delay. Even though the control design introduced is inspired by [17] (and [20]), the feedback law developed here is new. In particular, we properly define the quantizer's properties in an infinite-dimensional setting to account for the norm that the system under investigation is equipped with, as well as we properly choose the parameters of the adjustable parameter of the quantizer based on open- and closed-loop solutions' estimates that we derive for the class of systems considered. We establish global asymptotic stability in L^2 norm of the PDE state and in L^∞ norm for the actuator state. The stability analysis relies on a combination of input-to-state stability and small-gain arguments, with constructive derivation of solutions' estimates.

Section II introduces the class of systems and quantizers considered, along with the switched predictor-feedback design. Section III and Section IV establish global asymptotic stability of the closed-loop system under state and input quantization, respectively. Section V provides concluding remarks and suggests potential future research directions.

Notation: We denote by $L^2(0,1)$ the equivalence class of Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ such that $\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2} < \infty$. For a given function $u \in L^\infty([0,D];\mathbb{R})$ we define $\|u\|_\infty = \text{ess sup}_{x\in[0,D]} |u(x)|$ where ess sup, is the essential supremum. The state space $L^2([0,1];\mathbb{R})\times L^\infty([0,1];\mathbb{R})$ is induced with norm $\|(u,v)\| =$

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$\|u\|_2 + \|v\|_\infty$. We denote by $\mathcal{C}_{\rm rpw}(I;\mathbb{R})$ the space of right piecewise continuous functions $f : I \to \mathbb{R}$ (see [13]). For a given $h \in \mathbb{R}$ we define its integer part as $\lfloor h \rfloor = \max\{k \in \mathbb{Z} : k \le h\}$.

II. Problem Formulation and Control Design

A. Backstepping Control of Reaction-Diffusion PDE With Input Delay

Let us consider the following scalar reaction-diffusion PDE with known constant input delay $D > 0$

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) + \lambda u(t,x), && (1)\\ u(t,0) &= 0, && (2)\\ u(t,1) &= U(t-D), && (3)\end{aligned}$$

where $\lambda > \pi^2$ such that the open-loop system (1)–(3) is unstable. We pose this delay problem as an actuated transport PDE (modeling the delay phenomenon) which cascades into the boundary of the reaction-diffusion PDE,

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) + \lambda u(t,x), && (4)\\ u(t,0) &= 0, && (5)\\ u(t,1) &= v(t,0), && (6)\\ v_t(t,x) &= \tfrac{1}{D} v_x(t,x), && (7)\\ v(t,1) &= U(t), && (8)\end{aligned}$$

where $(t,x) \in \mathbb{R}_+ \times [0,1]$, $u(t,\cdot)$ and $v(t,\cdot)$ are respectively, the reaction-diffusion PDE and the transport PDE states at time t, with initial conditions $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$, $x \in [0,1]$, and variable $U(t)$ is control input.

Consider the following backstepping transformations

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy, \qquad (9)$$

$$z(t,x) = v(t,x) - D\int_0^x g(x,y)v(t,y)dy - \int_0^1 \gamma(x,y)u(t,y)dy, \qquad (10)$$

$x \in [0,1]$, with $\gamma(x,y)$, $k(x,y)$, and $g(x,y)$ given by

$$\gamma(x,y) = 2\sum_{n=1}^{\infty} e^{D(\lambda - n^2\pi^2)x}\sin(n\pi y) \times \int_0^1 \sin(n\pi\zeta)k(1,\zeta)d\zeta, \qquad (11)$$

with

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, \qquad (12)$$

on $\mathcal{T} := \{(x,y) : 0 \le y \le x \le 1\}$, where $\mathrm{I}_1(\cdot)$ denotes the modified Bessel function of first kind. In addition

$$g(x,y) = -\gamma_y(x-y,1). \qquad (13)$$

System (4)–(8) is transformed into the following system:

$$\begin{aligned} w_t(t,x) &= w_{xx}(t,x), && (14)\\ w(t,0) &= 0, && (15)\\ w(t,1) &= z(t,0), && (16)\\ z_t(t,x) &= \tfrac{1}{D} z_x(t,x), && (17)\\ z(t,1) &= d(t), && (18)\end{aligned}$$

with initial conditions

$$w_0(x) = u_0(x) - \int_0^x k(x,y)u_0(y)dy, \qquad (19)$$

$$z_0(x) = v_0(x) - \int_0^1 \gamma(x,y)u_0(y)dy - D\int_0^x g(x,y)v_0(y)dy, \qquad (20)$$

and the deviation $d(t)$ is defined by

$$d(t) =: U(t) - U_{\rm nom}(t), \qquad (21)$$

where $U_{\rm nom}(t)$ is the nominal predictor-feedback law

$$U_{\rm nom}(t) = \int_0^1 \gamma(1,y)u(t,y)dy + D\int_0^1 g(1,y)v(t,y)dy. \qquad (22)$$

The inverse transformation is given by

$$u(t,x) = w(t,x) + \int_0^x l(x,y)w(t,y)dy, \qquad (23)$$

$$v(t,x) = z(t,x) + \int_0^1 \delta(x,y)w(t,y)dy + D\int_0^x p(x,y)z(t,y)dy, \qquad (24)$$

with

$$l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, \qquad (25)$$

on $\mathcal{T} := \{(x,y) : 0 \le y < x \le 1\}$ where $\mathrm{J}_1(\cdot)$ denotes the Bessel function of first kind and

$$\delta(x,y) = 2\sum_{n=1}^{\infty} e^{-Dn^2\pi^2 x}\sin(n\pi y)\int_0^1 \sin(n\pi\zeta)l(1,\zeta)d\zeta, \qquad (26)$$

$$p(x,y) = -\delta_y(x-y,1). \qquad (27)$$

Using the estimates of the backstepping transformations, i.e.,

$$\begin{aligned} &\|w(t)\|_2 \le \tilde{k}\|u(t)\|_2, && (28)\\ &\|u(t)\|_2 \le \tilde{l}\|w(t)\|_2, && (29)\\ &\|z(t)\|_\infty \le \tilde{\gamma}\|u(t)\|_2 + \tilde{g}\|v(t)\|_\infty, && (30)\\ &\|v(t)\|_\infty \le \tilde{\delta}\|w(t)\|_2 + \tilde{p}\|z(t)\|_\infty, && (31)\end{aligned}$$

with $\tilde{k} := 1 + \left(\int_0^1\left(\int_0^x |k(x,y)|^2 dy\right)dx\right)^{1/2}$, $\tilde{l} := 1 + \left(\int_0^1\left(\int_0^x |l(x,y)|^2 dy\right)dx\right)^{1/2}$, $\tilde{g} := 1 + D\max_{0\le x\le 1}\int_0^x |g(x,y)|dy$, $\tilde{p} := 1 + D\max_{0\le x\le 1}\int_0^x |p(x,y)|dy$,

$\tilde{\gamma} := \sup_{x\in[0,D]} (\|\gamma(x,\cdot)\|_2)$, and $\tilde{\delta} := \sup_{x\in[0,D]} (\|\delta(x,\cdot)\|_2)$ one establishes the following inequality

$$M_2\|(u,v)\| \leq \|(w,z)\| \leq M_1\|(u,v)\|, \tag{32}$$

where M_1, M_2 are

$$M_1 = \max\{(\tilde{k}+\tilde{\gamma}), \tilde{q}\}, \quad M_2 = \tfrac{1}{\max\{(\tilde{l}+\tilde{\delta})\}}. \tag{33}$$

Although (9)–(33) are well-known facts, we present them here as the constants M_1 and M_2 are incorporated in the control design.

B. Properties of the Quantizer

The state u of the plant and the actuator state v are available only in quantized form. We consider here dynamic quantizers with an adjustable parameter of the form (see, e.g., [3], [17], [22])

$$q_\mu(u,v) = (q_{1\mu}(u), q_{2\mu}(v)) = \left(\mu q_1\left(\frac{u}{\mu}\right), \mu q_2\left(\frac{v}{\mu}\right)\right), \tag{34}$$

where $\mu > 0$ can be manipulated and this is called "zoom" variable. The quantizer $q_1 : L^2([0,1];\mathbb{R}) \to L^2([0,1];\mathbb{R})$ is Lipschitz on bounded sets (see, e.g. [31]), while $q_2 : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. Both quantizers satisfy the following properties

P1: If $\|(u,v)\| \leq M$, then $\|(q_1(u)-u, q_2(v)-v)\| \leq \Delta$,
P2: If $\|(u,v)\| > M$, then $\|(q_1(u), q_2(v))\| > M-\Delta$,
P3: If $\|(u,v)\| \leq \hat{M}$, then $q_1(u) = 0$[1] and $q_2(v) = 0$,
for some positive constants $M, \hat{M}$, and Δ, with $M > \Delta$ and $\hat{M} < M$.

Considering the L^2-norm of the reaction-diffusion PDE state requires to define the properties of the quantizer accordingly in L^2-norm, as well as to impose the additional assumption that the quantizer function also belongs to L^2 (that is tacitly assumed in the definition of q_1). An example of a quantizer that satisfies this property is when the quantizer may be assumed to also satisfy a specific sector condition (such as, e.g., the logarithmic quantizer [9]). In order to actually implement a quantizer using typical quantizer functions (see, e.g., [22]), when the quantizer is defined using an L^2-norm, one would have to also use, e.g., a density argument, to guarantee the approximation of its L^2-norm by the Euclidean norm of a finite-dimensional counterpart. The work in [30] is relevant here, as it introduces and implements quantizers in different norms. We also note that we choose to define the properties of the quantizer in terms of the norm of the complete infinite-dimensional state of the PDE-PDE system, as the system considered is equipped with this norm, in correspondence with respective definitions in the finite-dimensional case, see, e.g., [22].

We further notice that we consider a case in which the measured transport PDE state is also subject to quantization. If the transport PDE state corresponds to a delayed state and if the delayed state is available without quantization, one could employ the past input values in the predictor-feedback law. Moreover, as in [17], we assume a single tunable parameter μ for simplicity in control design and analysis, which is also practically reasonable in scenarios where, for example, a single computer with a single camera is used for measurements collection.

C. Predictor-Feedback Law Using Quantized Measurements

The hybrid predictor-feedback law is defined as

$$U(t) = \begin{cases} 0, & 0 \leq t < t_0 \\ P_{\mu(t)}\left(u(\cdot,t), v(\cdot,t)\right), & t \geq t_0 \end{cases}, \tag{35}$$

where

$$P_\mu(u,v) = \int_0^1 \gamma(1,y) q_{1\mu}(u(y))dy + D\int_0^1 g(1,y) q_{2\mu}(v(y))dy. \tag{36}$$

The tunable parameter μ is selected as

$$\mu(t) = \begin{cases} \overline{M}_1 \mathrm{e}^{2\sigma_1(j+1)\tau}\mu_0, & (j-1)\tau \leq t < j\tau + \bar{\tau}\delta_j, \\ & 1 \leq j \leq \left\lfloor \frac{t_0}{\tau} \right\rfloor, \\ \mu(t_0), & t \in [t_0, t_0+T), \\ \Omega\mu(t_0+(i-2)T), & t \in [t_0+(i-1)T, \\ & t_0+iT), \quad i = 2,3,\ldots \end{cases}, \tag{37}$$

for some fixed, yet arbitrary, $\tau, \mu_0 > 0$, where $t_0 = m\tau + \bar{\tau}$, for an $m \in \mathbb{Z}_+$, $\bar{\tau} \in [0,\tau)$, and $\delta_m = 1, \delta_j = 0, j < m$, with t_0 being the first time instant at which the following holds[2]

$$\left\|\mu(t_0) q_1\left(\frac{u(t_0)}{\mu(t_0)}\right)\right\|_2 + \left\|\mu(t_0) q_2\left(\frac{v(t_0)}{\mu(t_0)}\right)\right\|_\infty \leq (M\overline{M} - \Delta)\mu(t_0), \tag{38}$$

where

$$M_3 = \|\gamma(1,\cdot)\|_2 + D\max_{0\leqslant y\leqslant 1}|g(1,y)|, \tag{39}$$

$$\overline{M} = \tfrac{M_2}{M_1(1+M_0)}, \tag{40}$$

$$\overline{M}_1 = \max\{\sqrt{2}, G+1\}, \tag{41}$$

$$G^3 = 4\sqrt{\sum_{n=0}^{\infty} \tfrac{n^2\pi^2}{(\lambda - n^2\pi^2)^2}}, \tag{42}$$

$$\sigma_1 = \lambda - \pi^2, \tag{43}$$

$$\Omega = \frac{(1+\lambda_1)(1+M_0)^2\Delta M_3}{M_2 M}, \tag{44}$$

$$T = -\frac{\ln\left(\frac{\Omega}{1+M_0}\right)}{\delta}. \tag{45}$$

The parameters δ, λ_1 and M_0 are defined as follows. Parameter $\delta \in (0, \min\{\pi^2, \nu\})$, λ_1 is selected large enough

[1]The equality $q_1(u) = 0$ in P3 is understood in the sense of distributions.

[2]Note that (38) can be verified using only the availabe quantized measurements.

[3]For simplicity of derivations within the proof of Lemma 1 it is tacitly assumed that $\lambda \neq n^2\pi^2$ for all n. If it happens that $\lambda = \pi^2\bar{n}^2$ for some $\bar{n}$, one could show (with additional, but tedious computations) that (63) still holds with $\bar{M}_1 = \max\{\sqrt{2}, G+1\}$ and $G = 4\sqrt{\sum_{n\neq\bar{n}} \frac{n^2\pi^2}{(\lambda-n^2\pi^2)^2}} + 2\sqrt{2}\pi\bar{n}$.

in such a way that the following small-gain condition holds

$$\frac{1}{1+\lambda_1} < \frac{e^{-D}}{1+\frac{\sqrt{3}}{3}}, \tag{46}$$

and M_0 is defined such that

$$M_0 = (1-\varphi_1)^{-1}\max\left\{1; \tfrac{1}{\sqrt{3}}(1+\varepsilon)(1-\phi)^{-1}e^{D(\nu+1)}\right\} + (1-\phi)^{-1}(1-\varphi_1)^{-1}\max\left\{e^{D(\nu+1)}; \phi\right\}, \tag{47}$$

where $0<\phi<1$ and $0<\varphi_1<1$ with

$$\phi = \frac{1+\varepsilon}{1+\lambda_1}e^{D(\nu+1)} \text{ and } \varphi_1 = \tfrac{1}{\sqrt{3}}(1+\varepsilon)(1-\phi)^{-1}\phi, \tag{48}$$

for some $\varepsilon, \nu > 0$. The choice of ν, ε is such that it guarantees that $\phi < 1, \varphi_1 < 1$, which is always possible given (46).

III. Stability of Switched Predictor-Feedback Controller Under State Quantization

Theorem 3.1: Consider the closed-loop system consisting of the plant (4)–(8) and the switched predictor-feedback law (35)–(37). If Δ and M satisfy

$$\frac{\Delta}{M} < \frac{M_2}{(1+M_0)\max\{M_3(1+\lambda_1)(1+M_0), 2M_1\}}, \tag{49}$$

then for every initial data $v_0 \in C_{\mathrm{rpw}}([0,1];\mathbb{R})$ and $u_0 \in L^2(0,1)$, the solution (u,v) to (4)–(8) satisfies the property

$$\|u(t)\|_2 + \|v(t)\|_\infty \leq \gamma\left(\|u_0\|_2 + \|v_0\|_\infty\right)^{\left(2-\frac{\ln\Omega}{T}\frac{1}{\sigma_1}\right)} \mathrm{e}^{\frac{\ln\Omega}{T}t}, \tag{50}$$

where

$$\gamma = \frac{\overline{M}_1}{M_2}\max\left\{\frac{M_2 M}{\Omega}e^{2\sigma_1\tau}\mu_0, M_1\right\} \times \max\left\{\frac{1}{\mu_0(M\overline{M}-2\Delta)}, 1\right\} \times \left(\frac{1}{\mu_0(M\overline{M}-2\Delta)}\right)^{\left(1-\frac{\ln\Omega}{T}\frac{1}{\sigma_1}\right)}. \tag{51}$$

Although we do not state an existence and uniqueness result, as its proof is out of the scope of the present paper, which focuses on the control design and stability analysis, in principle, it can be studied as follows. Within the interval $[0,t_0)$, explicit solutions (thanks to spectral analysis and the method of characteristics), together with [14, Corollary 2.2], can be used to show that for every $v_0 \in C_{\mathrm{rpw}}([0,1])$ and $u_0 \in L^2(0,1)$, the solution to the open-loop system satisfies $u \in C\left([0,t_0); L^2(0,1)\right)$ and $v \in C_{rpw}([0,t_0)\times[0,1])$).

The well-posedness of the system (4)–(8) and (35) for $t \geq t_0$ can be proved by induction, starting in the interval $[t_0, t_0+T)$ (where μ is constant) as key step. Thanks to the Lipschitzness of the mapping $\varphi_\mu : L^2([0,1];\mathbb{R}) \times L^\infty([0,1];\mathbb{R}) \to \mathbb{R}$ defined as $\varphi_\mu(z,w) = \int_0^1 \gamma(1,y)\mu q_1\left(\frac{z(y)}{\mu}\right)dy + D\int_0^1 g(1,y)\mu q_2\left(\frac{w(y)}{\mu}\right)dy$ (given the Lipschitzness assumptions on q_1 and q_2), and similarly to the proof of [15, Theorem 11.3] we can apply Banach's fixed-point theorem (constructing proper mappings based on explicit solutions to (4)–(8)) to obtain a unique (local) solution to (4)–(8) and (35) satisfying $u \in C\left([t_0, t_0+T_1); L^2(0,1)\right)$ and $v \in \mathcal{C}_{\mathrm{rpw}}([t_0,t_0+T_1)\times[0,1])$ for some $0 < T_1 < T$. The (global) well-posedness on $[t_0, t_0+T)$ follows from the boundedness of the solution (u,v) in $L^2 \times L^\infty$ (thanks to Lemma 1 and Lemma 2). By induction, we obtain existence and uniqueness of a solution $u \in C\left(\mathbb{R}_+; L^2(0,1)\right)$ and $v \in \mathcal{C}_{\mathrm{rpw}}(\mathbb{R}_+ \times [0,1])$.

The following lemmas provide critical bounds on both the open-loop and closed-loop phases of the system, which are essential for establishing stability of the solutions.

Lemma 1: Let Δ and M satisfy (49), there exists a time t_0 satisfying

$$t_0 \leqslant \frac{1}{\sigma_1}\ln\left(\frac{\frac{1}{\mu_0}\left(\|u_0\|_2 + \|v_0\|_\infty\right)}{(M\overline{M}-2\Delta)}\right), \tag{52}$$

such that (38) holds, and thus, the following also holds

$$\|u(t_0)\|_2 + \|v(t_0)\|_\infty \leq M\overline{M}\mu(t_0). \tag{53}$$

Proof: For all $0 \leq t < t_0$, thanks to (35) one has $U(t) = 0$, and thus, the corresponding open-loop system reads as

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) + \lambda u(t,x), && (54)\\ u(t,0) &= 0, && (55)\\ u(t,1) &= v(t,0), && (56)\\ v_t(t,x) &= \tfrac{1}{D}v_x(t,x), && (57)\\ v(t,1) &= 0. && (58) \end{aligned}$$

The solution to the transport subsystem (57), (58), using the method of characteristics is given as $v(t,x) = v_0(\frac{1}{D}t + x)$ for $t \leq D(1-x)$ and $v(t,x) = 0$ for $t \geq D(1-x)$. Therefore,

$$\|v(t)\|_\infty \leqslant \|v_0\|_\infty. \tag{59}$$

From the equations (54)–(56) we use [12, Identity (3.11)] for $0 \leq t < t_0$, $\|u(t)\|_2 = \sqrt{\sum_{n=1}^\infty |c_n(t)|^2}$ with $c_n(t) = e^{\sigma_n t}c_n(0) - \frac{d\phi_n(1)}{dx}\int_0^t e^{\sigma_n(t-s)}v(s,0)ds$ and

$$|c_n(t)| \leq e^{\sigma_n t}|c_n(0)| + \left|\frac{d}{dx}\phi_n(1)\right|\frac{|1-e^{\sigma_n t}|}{\sigma_n} \times \text{ess sup}_{0\leqslant s\leqslant t}|v(s,0)|, \tag{60}$$

where $\sigma_n = \lambda - \pi^2 n^2$, $\phi_n(x) = \sqrt{2}\sin(\pi n x), n = 1,2,\ldots$. Applying the Young's inequality and using the inequalities $e^{\sigma_n t} = e^{\lambda t}e^{-\pi^2 n^2 t} \leq e^{\lambda t}e^{-\pi^2 t} = e^{\sigma_1 t}$ and $|1-e^{\sigma_n t}| \leq 1 + e^{\sigma_n t} \leq 1 + e^{\sigma_1 t} \leq 2e^{\sigma_1 t}$ we obtain

$$|c_n(t)|^2 \leqslant 2e^{2\sigma_1 t}|c_n(0)|^2 + \frac{8}{\sigma_n^2}\left|n\pi\sqrt{2}\cos(\pi n)\right|^2 \times e^{2\sigma_1 t}\left(\text{ess sup}_{0\leqslant s\leqslant t}|v(s,0)|\right)^2 \tag{61}$$

Therefore,

$$\|u(t)\|_{L^2} \leq \sqrt{2}e^{\sigma_1 t}\|u_0\|_2 + Ge^{\sigma_1 t}\|v_0\|_\infty, \tag{62}$$

where G and σ_1 are given by (41) and (42) respectively. Therefore, combining (59) and (62) we have for $0 \leq t < t_0$

$$\|u(t)\|_2 + \|v(t)\|_\infty \leqslant \overline{M}_1 e^{\sigma_1 t} \left(\|u_0\|_2 + \|v_0\|_\infty\right), \quad (63)$$

with $\overline{M}_1$ defined in (43). Choosing the switching signal μ according to (37), one has the existence of a time t_0 verifying (52) such that

$$\frac{\|u\left(t_0\right)\|_2}{\mu(t_0)} + \frac{\|v\left(t_0\right)\|_\infty}{\mu(t_0)} \leq M\overline{M} - 2\Delta. \quad (64)$$

The rest of the proof is identical to that of Lemma 5 in [18]. ∎

Lemma 2: Select λ_1 large enough in such a way that the small-gain condition (46) holds. Then the solutions to the target system (14)–(20) with the quantized controller (35), which verify, for fixed μ,

$$\|w(t_0)\|_2 + \|z(t_0)\|_\infty \leq \frac{M_2}{1+M_0} M\mu, \quad (65)$$

they satisfy for $t_0 \leq t < t_0 + T$

$$\|w(t)\|_2 + \|z(t)\|_\infty \leqslant \max\left\{M_0 e^{-\delta(t-t_0)}\left(\|w(t_0)\|_2 + \|z(t_0)\|_\infty\right), \Omega\frac{M_2}{1+M_0}M\mu\right\}. \quad (66)$$

In particular, the following holds

$$\|w(t_0+T)\|_2 + \|z(t_0+T)\|_\infty \leq \Omega\frac{M_2}{1+M_0}M\mu. \quad (67)$$

Proof: See the proof of Lemma 6 in [18]. ∎

Proof of Theorem 3.1: See the proof of Theorem 3 in [18]. □

IV. Extension to Input Quantization

The results in the previous section could be extended to input quantization by modifying the switched predictor-feedback law as follows

$$U(t) = \begin{cases} 0, & 0 \leq t < \bar{t}_0 \\ \bar{q}_\mu\left(U_{\mathrm{nom}}(t)\right), & t \geq \bar{t}_0 \end{cases}, \quad (68)$$

where $U_{\mathrm{nom}}(t)$ is given in (22) and the quantizer is a locally Lipschitz function $\bar{q}_\mu : \mathbb{R} \to \mathbb{R}$, defined by $\bar{q}_\mu(\bar{U}) = \mu\bar{q}\left(\frac{\bar{U}}{\mu}\right)$, satisfying the following properties

$\bar{\mathrm{P}}1$: If $|\bar{U}| \leq M$, then $|\bar{q}(\bar{U}) - \bar{U}| \leq \Delta$,

$\bar{\mathrm{P}}2$: If $|\bar{U}| > M$, then $|\bar{q}(\bar{U})| > M - \Delta$,

$\bar{\mathrm{P}}3$: If $|\bar{U}| \leq \hat{M}$, then $\bar{q}(\bar{U}) = 0$.

Considering the parameters defined in (39)–(45), the switching variable μ is selected as

$$\mu(t) = \begin{cases} \overline{M}_1 \mathrm{e}^{2\sigma_1(j+1)\tau}\mu_0, & (j-1)\tau \leq t < j\tau + \bar{\tau}_1\delta_j, \\ & 1 \leq j \leq \left\lfloor \frac{\bar{t}_0}{\tau} \right\rfloor, \\ \mu\left(\bar{t}_0\right), & t \in [\bar{t}_0, \bar{t}_0 + T), \\ \Omega\mu\left(\bar{t}_0 + (i-2)T\right), & t \in [\bar{t}_0 + (i-1)T, \\ & \bar{t}_0 + iT), \quad i = 2, 3, \ldots \end{cases}, \quad (69)$$

for some fixed, yet arbitrary, $\tau, \mu_0 > 0$, where $\bar{t}_0 = \bar{m}\tau + \bar{\tau}_1$, for an $\bar{m} \in \mathbb{Z}_+$, $\bar{\tau}_1 \in [0, \tau)$, and $\delta_{\bar{m}} = 1, \delta_j = 0, j < \bar{m}$, with $\bar{t}_0$ being the first time instant at which the following event is detected using the available measurements of the actuators states are available, holds

$$\|u(\bar{t}_0)\|_2 + \|v(\bar{t}_0)\|_\infty \leq \frac{M\overline{M}}{M_3}\mu(\bar{t}_0). \quad (70)$$

Theorem 4.1: Consider the closed-loop system consisting of the plant (4)–(8) and the switched predictor-feedback law (68), (69) with (22). If Δ and M satisfy

$$\frac{\Delta}{M} < \frac{M_2}{M_3(1+\lambda_1)(1+M_0)^2}, \quad (71)$$

then for every initial data $v_0 \in C_{\mathrm{rpw}}([0,1];\mathbb{R})$ and $u_0 \in L^2(0,1)$ the solution (u, v) to (4)–(8) satisfies

$$\|u(t)\|_2 + \|v(t)\|_\infty \leq \bar{\gamma}\left(\|u_0\|_2 + \|v_0\|_\infty\right)^{\left(2 - \frac{\ln\Omega}{T}\frac{1}{\sigma_1}\right)} \mathrm{e}^{\frac{\ln\Omega}{T}t}, \quad (72)$$

where

$$\bar{\gamma} = \frac{\sigma_1}{M_2}\max\left\{\frac{M_2 M}{\Omega M_3}e^{2\sigma_1\tau}\mu_0, M_1\right\}\max\left\{\frac{M_3}{\mu_0 M\overline{M}}, 1\right\} \times \left(\frac{M_3}{\mu_0 M\overline{M}}\right)^{\left(1 - \frac{\ln\Omega}{T}\frac{1}{\sigma_1}\right)}. \quad (73)$$

To prove Theorem 4.1, we first establish two lemmas, whose proofs employ reasoning similar to that used in the case of state quantization.

Lemma 3: There exists a time $\bar{t}_0$ satisfying

$$\bar{t}_0 \leqslant \frac{1}{\sigma_1}\ln\left(\frac{\frac{M_3}{\mu_0}\left(\|u_0\|_2 + \|v_0\|_\infty\right)}{M\overline{M}}\right), \quad (74)$$

such that (70) holds.

Proof: For all $0 \leq t < \bar{t}_0$, the system is described by (54)–(58).Thus, we obtain, as in the proof of Lemma 1, estimate (63). By selecting the switching signal μ according to (69), there exists a time $\bar{t}_0$ that satisfies (74), ensuring that the relation (70) holds. ∎

Lemma 4: Select λ_1 large enough in such a way that the small-gain condition (46) holds. Then, the solutions to the target system (14)–(18) with the quantized controller (22), (68), (69), which verify, for fixed μ,

$$\|w(\bar{t}_0)\|_2 + \|z(\bar{t}_0)\|_\infty \leq \frac{M_2 M\mu}{(1+M_0)M_3}, \quad (75)$$

they satisfy for $\bar{t}_0 \leq t < \bar{t}_0 + T$

$$\|w(t)\|_2 + \|z(t)\|_\infty \leqslant \max\left\{M_0 e^{-\delta(t-\bar{t}_0)}\left(\|w(\bar{t}_0)\|_2 + \|z(\bar{t}_0)\|_\infty\right), \frac{\Omega M_2 M\mu}{(1+M_0)M_3}\right\}. \quad (76)$$

In particular, the following holds

$$\|w(\bar{t}_0+T)\|_2 + \|z(\bar{t}_0+T)\|_\infty \leq \frac{\Omega M_2 M\mu}{(1+M_0)M_3}. \quad (77)$$

Proof: Using the same strategy, as in the proof of Lemma 2 the following inequalities hold

$$\|w\|_{[\bar{t}_0,t]} \leq \|w(\bar{t}_0)\|_2 + \tfrac{1}{\sqrt{3}}(1+\varepsilon)\|z\|_{[\bar{t}_0,t]}, \quad (78)$$

$$\|z\|_{[\bar{t}_0,t]} \leq e^{D(\nu+1)}\|z(\bar{t}_0)\|_\infty + e^{D(\nu+1)} \times (1+\varepsilon)\operatorname{ess\,sup}_{\bar{t}_0 \leqslant s \leqslant t}\left(\left|\bar{d}(s)\right| e^{\delta(s-\bar{t}_0)}\right), \quad (79)$$

where for U_{nom} and μ given in (22) and (69), respectively,

$$\bar{d}(t) = \mu(t)\bar{q}\left(\frac{U_{\text{nom}}(t)}{\mu(t)}\right) - U_{\text{nom}}(t). \tag{80}$$

Provided that

$$\frac{\Omega}{(1+M_0)^2}\frac{M_2}{M_3}M\mu \leq \|w\|_2 + \|z\|_\infty \leqslant \frac{M_2}{M_3}M\mu, \tag{81}$$

using the property $\bar{\text{P}}1$ of the quantizer, the left-hand side of bound (32), and the definition (44), we obtain from (80)

$$\left|\bar{d}\right| \leqslant \frac{1}{1+\lambda_1}\left(\|w\|_2 + \|z\|_\infty\right).$$

Repeating the respective arguments from the proof of Lemma 2, we arrive at

$$\|w(t)\|_2 + \|z(t)\|_\infty \leqslant M_0 e^{-\delta(t-\bar{t}_0)}\left(|X(\bar{t}_0)| + \|w(\bar{t}_0)\|_\infty\right). \tag{82}$$

Thus, for $\bar{t}_0 \leq t < \bar{t}_0 + T$, using relation (75) we get

$$\|w(t)\|_2 + \|z(t)\|_\infty \leq \frac{M_2}{M_3}M\mu, \tag{83}$$

which makes estimate (82) legitimate. Moreover, thanks to relation (75) and (45), one obtains from (82)

$$\|w(\bar{t}_0+T)\|_2 + \|z(\bar{t}_0+T)\|_\infty \leqslant \frac{\Omega M_2 M\mu}{(1+M_0)M_3}, \tag{84}$$

and hence, bound (77) is obtained. The rest of the proof utilizes the same reasoning as the proof of Lemma 2. ■

Proof of Theorem 4.1: The proof of Theorem 4.1 follows a similar approach to that used in the corresponding part of the proof of Theorem 3.1, with Lemmas 3 and 4 playing analogous roles to Lemmas 1 and 2.

V. Conclusions and Future Work

In this paper, we achieved the global asymptotic stability of an unstable reaction-diffusion PDE with input delay under both state and input quantization, developing a switched predictor-feedback control law and combining the backstepping method with small-gain and input-to-state stability techniques. Future research could explore designing event-triggered controllers under quantization, inspired by [19]. The challenges for such a design would be the construction of the event-triggering mechanism such that it depends only on quantized measurements, as well as the proof of Zeno phenomenon avoidance, due to the non-differentiability of the quantizer function.

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