Simultaneous Compensation of Input Delay and State Quantization for Linear Systems via Switched Predictor Feedback

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Abstract— **We develop a switched predictor-feedback law, which achieves global asymptotic stabilization of linear systems with input delay and with the plant and actuator states available only in (almost) quantized form. The control design relies on a quantized version of the nominal predictor-feedback law for linear systems, in which quantized measurements of the plant and actuator states enter the predictor state formula. A switching strategy is constructed to dynamically adjust the tunable parameter of the quantizer (in a piecewise constant manner), in order to initially increase the range and subsequently decrease the error of the quantizers. The key element in the proof of global asymptotic stability in the supremum norm of the actuator state is derivation of solutions' estimates combining a backstepping transformation with small-gain and input-to-state stability arguments, for addressing the error due to quantization.**

I. Introduction

Compensation of long input delays for linear systems can be achieved via predictor-based control design techniques and, in particular, via exact predictor feedbacks, see, for example, [1], [4], [7], [18]. The baseline, continuous predictor-based designs are accompanied with certain stability (and robustness) guarantees, see, e.g., [4], [5], [14], [18]. However, implementation of predictor feedbacks may be subject to digital effects, such as, for example, sampling and quantization, which may deteriorate closed loop performance of the nominal continuous designs, when these effects are left uncompensated, see, for example, [14], [20]. Therefore, addressing issues arising due to digital implementation of predictor feedbacks is practically and theoretically significant, in order to provably preserve the stability guarantees of the original, continuous designs.

Among the potential digital implementation issues arising in implementation of predictor feedbacks, sampled measurements and control inputs applied via zero-order hold have been addressed in [13], [24], [25]; while [2], [27] address sampling in sequential predictors-based designs and [8], [10] address sampling in nonlinear systems with state delay. In particular, [13], [24], [25] introduce design and analysis approaches for compensating the effect of sampling in measurements and actuation employing predictor-based designs; while [9] also addresses quantization effects in nonlinear systems with delay on the state. Because our design also relies on a switched strategy (although it is assumed that the input is continuously applied and that continuous measurements are available), results on predictor-based event-triggered control design may be also viewed as relevant [12], [20], [21], [22], [26]. Besides [3] that considers the case of a class of boundary controlled, first-order hyperbolic PDEs (Partial Differential Equations) subject to state quantization, not involving an ODE (Ordinary Differential Equation) part, paper [23] presents a predictor-feedback design for a particular case of a model of robot manipulator with quantized input, which, however, neither addresses a general linear system with a systematic design and analysis approach nor aims at achieving asymptotic stability via a dynamic quantizer, while quantization affects the control input and not the state measurements.

In the present paper, we develop a switched predictor-feedback control design that achieves compensation of both input delay and state quantization. The control design relies on two main ingredients–a baseline predictor-feedback design and a switching strategy that dynamically adjusts the tunable parameter of the quantizer. In particular, the nominal predictor state formula is modified such that the plant and actuator states are replaced by their quantized versions; while the tunable parameter of the quantizer is dynamically adjusted (in a piecewise manner), in order to initially increase the range of the quantizer and to subsequently decreases its error, as it is done in the case of delay-free systems in [6], [19]. The transition between the two main modes of operation of the switching signal takes place at detection of an event indicating that the infinite-dimensional state of the system (consisting of the ODE and actuator states) enters the range of the quantizer, implying that stabilization can be then achieved decreasing the quantization error.

We establish global asymptotic stability in the supremum norm of the actuator state. The proof strategy relies on combination of backstepping [18] with small-gain and input-to-state stability (ISS) arguments [16], towards derivation of estimates on solutions. In particular, the proof consists of two main steps. In the first, the system operates in open loop, while the range of the quantizer is increasing. Deriving estimates based on the explicit solution of the open-loop system, it is shown that there exists some time instant at which the state enters within the range of the quantizer. In the second step of the proof, given that the state is within the quantizer's range, the tunable parameter of the quantizer is decreasing in a piecewise constant manner. In particular, within each interval in which the quantizer's parameter is constant, we

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show that the norm of the solutions of the closed-loop system decreases by a factor that is less than unity. To show this we capitalize on the input-to-state stability properties of linear predictor feedbacks and of the pure-transport PDE system, which allow us to obtain, via a small-gain argument, a condition on the quantizer's parameters, namely, on its range and error, which guarantees that asymptotic stability is achieved. We note that the quantizers considered here are called "almost" quantizers. The reason is that we use functions that are locally Lipschitz and not just piecewise constant functions to avoid issues related to existence and uniqueness of solutions[1]. This assumption allows us to study existence and uniqueness using the results from [16] in combination with [11], [15], while the stability estimates derived and the overall proof strategy adopted do not depend on this property, suggesting that such an assumption can be removed.

We start in Section II presenting the classes of systems and quantizers considered, together with the switched predictor-feedback design. In Section III we establish global asymptotic stability of the closed-loop system. In Section IV we provide concluding remarks and a discussion on our ongoing research.

Notation: We denote by $L^\infty(A;\Omega)$ the space of measurable and bounded functions defined on A and taking values in Ω. For a given $D>0$ and a function $u\in L^\infty([0,D];\mathbb{R})$ we define $\|u\|_\infty = \text{ess sup}_{x\in[0,D]}|u(x,t)|$. For a given $h\in\mathbb{R}$ we define its integer part as $\lfloor h\rfloor = \max\{k\in\mathbb{Z}: k\le h\}$. The state space $\mathbb{R}^n\times L^\infty([0,D];\mathbb{R})$ is induced with norm $\|(X,u)\| = |X| + \|u\|_\infty$. We denote by $AC(\mathbb{R}_+,\mathbb{R}^n)$, the set of all absolutely continuous function $X:\mathbb{R}_+\to\mathbb{R}^n$. Let $I\subseteq\mathbb{R}$ be an interval. A piecewise left-continuous function $f: I\to J$ is a function continuous on each closed interval subset of I except possibly on a finite number of points $x_0 < x_1 < \cdots < x_p$ such that for all $l\in\{0,\cdots,p-1\}$ there exists f_l continuous on $[x_l,x_{l+1}]$ and $f_l|_{(x_l,x_{l+1})} = f|_{(x_l,x_{l+1})}$. Moreover, at the points $x_0,\cdots,x_p$ the function is left continuous. The set of all piecewise left-continuous functions is denoted by $\mathcal{C}_{lpw}(I,J)$ (see also [11], [15]).

II. Problem Formulation and Control Design

A. Linear Systems With Input Delay & State Quantization

We consider the following system

$$\dot X(t) = AX(t) + BU(t-D), \tag{1}$$

where $D>0$ is constant input delay, $t\ge 0$ is time variable, $X\in\mathbb{R}^n$ is state, and U is scalar control input. An alternative representation of this system is as follows

$$\dot X(t) = AX(t) + Bu(0,t), \tag{2}$$
$$u_t(x,t) = u_x(x,t), \tag{3}$$
$$u(D,t) = U(t), \tag{4}$$

by setting $u(x,t) = U(t+x-D)$, where $x\in[0,D]$ and u is the transport PDE state, with initial conditions $u(x,0) = u_0(x)$. We proceed from now on with representation (2)–(4) as it turns out to be more convenient for control design and analysis. In [18] system (2)–(4) is transformed into

$$\dot X(t) = (A+BK)X(t) + Bw(0,t), \tag{5}$$
$$w_t(x,t) = w_x(x,t), \tag{6}$$
$$w(D,t) = U(t) - U_{\rm nom}(t), \tag{7}$$

thanks to the backstepping transformation

$$w(x,t) = u(x,t) - K\int_0^x e^{A(x-y)}Bu(y,t)dy - Ke^{Ax}X(t), \tag{8}$$

where $U_{\rm nom}(t)$ is the nominal predictor feedback defined by

$$U_{\rm nom}(t) = K\int_0^D e^{A(D-y)}Bu(y,t)dy + Ke^{AD}X(t). \tag{9}$$

The inverse of this transformation is

$$u(x,t) = w(x,t) + K\int_0^x e^{(A+BK)(x-y)}Bw(y,t)dy + Ke^{(A+BK)x}X(t). \tag{10}$$

One has

$$M_2\|(X,u)\| \le \|(X,w)\| \le M_1\|(X,u)\|, \tag{11}$$

where M_1, M_2 are

$$M_1 = |K|\mathrm{e}^{|A|D}\max\{1,D|B|\}+1, \tag{12}$$
$$M_2 = \frac{1}{|K|\mathrm{e}^{|A+BK|D}\max\{1,D|B|\}+1}. \tag{13}$$

Although (8)–(13) are well-known facts, we present them here as the constants M_1 and M_2 are incorporated in the control design.

B. Properties of the Quantizer

The state X of the plant and the actuator state u are available only in quantized form. We consider here dynamic quantizers with an adjustable parameter of the form (see, e.g., [6], [19])

$$q_\mu(X,u) = (q_{1\mu}(X), q_{2\mu}(u)) = \left(\mu q_1\left(\frac{X}{\mu}\right), \mu q_2\left(\frac{u}{\mu}\right)\right), \tag{14}$$

where $\mu>0$ can be manipulated and this is called "zoom" variable. The quantizers $q_1:\mathbb{R}^n\to\mathbb{R}^n$ and $q_2: L^\infty([0,D];\mathbb{R})\to\mathbb{R}$ are locally Lipchitz functions that satisfy the following properties

P1:If $\|(X,u)\|\le M$, then $\|(q_1(X)-X, q_2(u)-u)\|\le\Delta$,
P2:If $\|(X,u)\| > M$, then $\|(q_1(X),q_2(u))\| > M-\Delta$,
P3:If $\|(X,u)\|\le\hat M$, then $q_1(X)=0$ and $q_2(u)=0$,

for some positive constants $M,\hat M$, and Δ, with $M>\Delta$. When the argument of the employed quantizer is a vector, the quantizer function is a vector itself, defined componentwise according to (14), satisfying properties P1–P3. In the present case, we consider uniform quantizers for each element of the vector argument, while we discern between the quantizer function of the measurements of the plant state X and the function that corresponds to the actuator state measurements

[1]Arising due to the potential non-measurability of composition of an exact quantizer function with an only continuous PDE state.

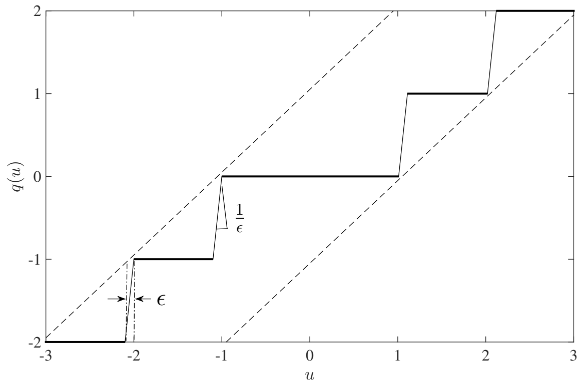

Fig. 1: An approximate quantizer with ϵ-layer.

u. For simplicity of control design and analysis we assume a single tunable parameter μ. In fact, this is also practically reasonable, considering a case where, e.g., a single computer with a single camera collects measurements. The quantizers considered here differ from typical piecewise constant quantizers, taking finitely many values, in that we assume they are locally Lipschitz functions. Although this may appear as a restrictive requirement, in practice, it isn't as it is also illustrated in Fig. 1, which shows a locally Lipschitz quantizer that may arbitrarily closely approximate a quantizer with rectilinear quantization regions. This is a technical requirement to guarantee existence and uniqueness of solutions in a straightforward manner. The stability result obtained and the stability proof do not essentially rely on this, suggesting that this assumption could be removed.

C. Predictor-Feedback Law Using Almost Quantized Measurements

The hybrid predictor-feedback law can be viewed as a quantized version of the predictor-feedback controller (9), in which the dynamic quantizer depends on a suitably chosen piecewise constant signal μ. It is defined as

$$U(t)=\begin{cases} 0, & 0\leq t\leq t_0 \\ KP_{\mu(t)}\left(X(t),u(\cdot,t)\right), & t>t_0 \end{cases}, \quad (15)$$

with

$$P_\mu(X,u)=\mathrm{e}^{AD}q_{1\mu}(X)+\int_0^D \mathrm{e}^{A(D-y)}Bq_{2\mu}(u(y))dy, \quad (16)$$

and K is a gain vector that makes matrix $A+BK$ Hurwitz. The tunable parameter μ is selected as[2]

$$\mu(t)=\begin{cases} \overline{M}_1\mathrm{e}^{2|A|(j+1)\tau}\mu_0, & (j-1)\tau\leq t\leq j\tau+\bar{\tau}\delta_j, \\ & 1\leq j\leq \left\lfloor \frac{t_0}{\tau}\right\rfloor, \\ \mu\left(t_0\right), & t\in(t_0,t_0+T], \\ \Omega\mu\left(t_0+(i-1)T\right), & t\in(t_0+(i-1)T, \\ & t_0+iT], \quad i=2,3,\ldots \end{cases}, \quad (17)$$

[2]In the particular case where $|A|=0$, one could replace $|A|$ by an arbitrary positive constant; while if $m=0$, then the first component of μ is applied for $t\in[0,\bar{\tau}]$ (with $j=0$).

for some fixed, yet arbitrary, $\tau,\mu_0>0$, where $t_0=m\tau+\bar{\tau}$, for an $m\in\mathbb{Z}_+$, $\bar{\tau}\in[0,\tau)$, and $\delta_m=1,\delta_j=0,j<m$, with t_0 being the first time instant at which the following holds

$$\left|\mu(t_0)q_1\left(\frac{X\left(t_0\right)}{\mu\left(t_0\right)}\right)\right|+\left\|\mu(t_0)q_2\left(\frac{u\left(t_0\right)}{\mu\left(t_0\right)}\right)\right\|_\infty \leq (M\overline{M}-\Delta)\mu(t_0), \quad (18)$$

where

$$M_3=|K|\mathrm{e}^{|A|D}(1+|B|D), \quad (19)$$

$$\overline{M}=\frac{M_2}{M_1(1+M_0)}, \quad (20)$$

$$\overline{M}_1=1+D|B|, \quad (21)$$

$$\Omega=\frac{(1+\lambda)(1+M_0)^2\Delta M_3}{M_2M}, \quad (22)$$

$$T=-\frac{\ln\left(\frac{\Omega}{1+M_0}\right)}{\delta}. \quad (23)$$

The parameters δ,λ and M_0 are defined as follows. Constant $\delta\in(0,\min\{\sigma,\nu\})$, for some $\nu,\sigma>0$, satisfying

$$\left|e^{(A+BK)t}\right|\leq M_\sigma e^{-\sigma t}, \quad (24)$$

for some $M_\sigma>1$, λ is selected large enough in such a way that the following small-gain condition holds

$$\frac{e^{D(\nu+1)}}{1+\lambda}\left(\frac{M_\sigma}{\sigma}|B|+1\right)<1, \quad (25)$$

and M_0 is defined such that

$$M_0=\max\left\{(1-\phi)^{-1}\left(1-\varphi_1\right)^{-1}e^{D(\nu+1)};\left(1-\varphi_1\right)^{-1}\phi \times M_\sigma\left(1-\phi\right)^{-1}\right\}+\max\left\{(1-\varphi_1)^{-1}M_\sigma;(1-\phi)^{-1} \times(1+\varepsilon)(1-\varphi_1)^{-1}e^{D(\nu+1)}\frac{M_\sigma}{\sigma}|B|\right\}, \quad (26)$$

where $0<\phi<1$ and $0<\varphi_1<1$ with

$$\phi=\frac{1+\varepsilon}{1+\lambda}e^{D(\nu+1)} \text{ and } \varphi_1=(1+\varepsilon)(1-\phi)^{-1}\phi\frac{M_\sigma}{\sigma}|B|, \quad (27)$$

for some $\varepsilon>0$. The choice of ν,ε guarantees that $\phi<1,\varphi_1<1$, which is always possible given (25) (see also the proof of Lemma 2 in Section III).

We note here few remarks for the control law presented. Event (18) can be detected using measurements of $q_{1\mu}(X),q_{2\mu}(u)$, and μ only, which are available. The tunable parameter μ is chosen in a piecewise constant manner. In the first phase (for $0\leq t\leq t_0$) it is increasing sufficiently fast, choosing a sufficiently large $\overline{M}_1$, so that there exists a time t_0 such that (18) holds (see Lemma 1) which depends on the size of open-loop solutions. In the zooming in phase, μ is decreasing by a factor of Ω over T time units. To guarantee convergence of the state to zero, Ω has to be less than one, which is guaranteed by assumption (see Theorem 1). Time T has to be chosen large enough, as T time units intervals correspond to intervals in which the solutions approach the equilibrium. Condtion (25) is a small-gain condition derived when viewing as disturbance the error due to quantized measurements and applying an ISS argument (see Lemma 2). In particular, when D increases, λ has to increase, which (via Ω in (22)) imposes stricter conditions on the ratio $\frac{\Delta}{M}$ between error and range of the quantizer.

III. Stability of Switched Predictor-Feedback Controller Under State Quantization

Theorem 1: Consider the closed-loop system consisting of the plant (2)–(4) and the switched predictor-feedback law (15)–(17). Let the pair (A, B) be stabilizable. If Δ and M satisfy

$$\frac{\Delta}{M} < \frac{M_2}{(1+M_0)\max\{M_3(1+\lambda)(1+M_0), 2M_1\}}, \quad (28)$$

then for all $X_0 \in \mathbb{R}^n$, $u_0 \in \mathcal{C}_{lpw}([0, D], \mathbb{R})$, there exists a unique solution such that $X(t) \in AC(\mathbb{R}_+, \mathbb{R}^n)$, for each $t \in \mathbb{R}_+$ $u(\cdot, t) \in \mathcal{C}_{lpw}([0, D], \mathbb{R})$, and for each $x \in [0, D]$ $u(x, \cdot) \in \mathcal{C}_{\text{lpw}}(\mathbb{R}_+, \mathbb{R})$, which satisfies

$$|X(t)| + \|u(t)\|_\infty \le \gamma \left(|X_0| + \|u_0\|_\infty\right)^{\left(2 - \frac{\ln \Omega}{T}\frac{1}{|A|}\right)} \mathrm{e}^{\frac{\ln \Omega}{T} t}, \quad (29)$$

where

$$\gamma = \frac{\overline{M}_1}{M_2} \max\left\{\frac{M_2 M}{\Omega} e^{2|A|\tau}\mu_0, M_1\right\} \max\left\{1, \frac{1}{\mu_0(M\overline{M} - 2\Delta)}\right\} \left(\frac{1}{\mu_0(M\overline{M} - 2\Delta)}\right)^{\left(1 - \frac{\ln \Omega}{T}\frac{1}{|A|}\right)}. \quad (30)$$

The proof relies on Lemmas 1 and 2 which are presented next. In particular, Lemma 1 establishes a bound on the solutions during the zooming out (open-loop) phase, which in turn is utilized to prove the existence of a time instant at which the solutions get within the range of the quantizer (ie., they satisfy (18)). Subsequently, Lemma 2 establishes a bound on closed-loop solutions, via employing a small-gain based ISS argument, for time intervals of constant μ. This bound implies that the solutions' magnitude decays by a factor of Ω every T time units.

Lemma 1: Let Δ and M satisfy (28), there exists a time t_0 satisfying

$$t_0 \leqslant \frac{1}{|A|} \ln\left(\frac{\frac{1}{\mu_0}\left(|X_0| + \|u_0\|_\infty\right)}{(M\overline{M} - 2\Delta)}\right), \quad (31)$$

such that (18) holds, and thus, the following also holds

$$|X(t_0)| + \|u(t_0)\|_\infty \le M\overline{M}\mu(t_0). \quad (32)$$

Proof: We refer to [17] for the proof of this Lemma. ■

Lemma 2: Choose K such that $A + BK$ is Hurwitz and let $\sigma, M_\sigma > 0$ be such that (24) holds. Select λ large enough in such a way that the small-gain condition (25) holds. Then the solutions to the target system (5)–(7) with the quantized controller (15), resulting in $w(D, t) = Ke^{AD}\mu(t)\left(q_1\left(\frac{X(t)}{\mu(t)}\right) - \frac{X(t)}{\mu(t)}\right) + K\int_0^D e^{A(D-y)}B\mu(t)\left(q_2\left(\frac{u(y,t)}{\mu(t)}\right) - \frac{u(y,t)}{\mu(t)}\right)dy$, with u given in terms of (X, w) by the inverse backstepping transformation (10), which verify, for fixed μ,

$$|X(t_0)| + \|w(t_0)\|_\infty \le \frac{M_2}{1+M_0} M\mu, \quad (33)$$

they satisfy for $t_0 < t \le t_0 + T$

$$|X(t)| + \|w(t)\|_\infty \leqslant \max\left\{M_0 e^{-\delta(t-t_0)}\left(|X(t_0)| + \|w(t_0)\|_\infty\right), \Omega\frac{M_2}{1+M_0}M\mu\right\}. \quad (34)$$

In particular, the following holds

$$|X(t_0+T)| + \|w(t_0+T)\|_\infty \le \Omega\frac{M_2}{1+M_0}M\mu. \quad (35)$$

Proof: Let $\varepsilon > 0$, using the fading memory lemma [16, Lemma 7.1], one can prove the existence of $\delta, \nu > 0$ s.t

$$|X|_{[t_0,t]} \leqslant M_\sigma|X(t_0)| + (1+\varepsilon)\frac{M_\sigma}{\sigma}|B|\|w\|_{[t_0,t]}, \quad (36)$$

and

$$\|w\|_{[t_0,t]} \le e^{D(\nu+1)}\|w(t_0)\|_\infty \quad (37)$$

$$+ e^{D(\nu+1)}(1+\varepsilon)\sup_{t_0\leqslant s\leqslant t}\left(|d(s)|e^{\delta(s-t_0)}\right), \quad (38)$$

with $d(t) = U(t) - U_{\text{nom}}(t)$, $\|w\|_{[t_0,t]} := \sup_{t_0\le s\le t}\left(\|w(s)\|_\infty e^{\delta(s-t_0)}\right)$ and $|X|_{[t_0,t]} := \sup_{t_0\leqslant s\leqslant t}|X(s)|e^{\delta(s-t_0)}$.

Let us next estimate the term $\sup_{t_0\leqslant s\leqslant t}\left(|d(s)|e^{\delta(s-t_0)}\right)$. For $t_0 < t \le t_0 + T$

$$|d| \le M_3\mu\left\|q_1\left(\frac{X}{\mu}\right) - \frac{X}{\mu}, q_2\left(\frac{u}{\mu}\right) - \frac{u}{\mu}\right\|, \quad (39)$$

with M_3 defined in (19) and u given in terms of (X, w) by the inverse backstepping transformation (10). Provided that

$$\Omega\frac{M_2}{(1+M_0)^2}M\mu \le |X| + \|w\|_\infty \leqslant M_2 M\mu, \quad (40)$$

thanks to the property P1 of the quantizer, the left-hand side of bound (11), and the definition (22), we obtain

$$|d| \le M_3\Delta\mu \leqslant \frac{1}{1+\lambda}\left(|X| + \|w\|_\infty\right). \quad (41)$$

Therefore, as long as the solutions satisfy (40) we get

$$\sup_{t_0\leqslant s\leqslant t}\left(|d(s)|e^{\delta(s-t_0)}\right) \leqslant \frac{1}{1+\lambda}\|w\|_{[t_0,t]} + \frac{1}{1+\lambda}|X|_{[t_0,t]}. \quad (42)$$

Combining estimate (36) with (38) and (42) we obtain the following estimate using the small-gain condition (25)

$$|X(t)| + \|w(t)\|_\infty \leqslant M_0 e^{-\delta(t-t_0)}\left(|X(t_0)| + \|w(t_0)\|_\infty\right), \quad (43)$$

with M_0 given in (26). For $t_0 < t \le t_0 + T$, using relation (33), the fact that $e^{-\delta(t-t_0)} \le 1$, and $\frac{M_0}{1+M_0} < 1$ one has that

$$|X(t)| + \|w(t)\|_\infty \le M_2 M\mu, \quad (44)$$

which makes estimate (43) legitimate. Moreover, at the time instant $t_0 + T$, thanks to the relation (33) and the definition

(23) of T, one obtains (34) from (43). Note that relations (35) and (43) are established provided that $\Omega\frac{M_2}{(1+M_0)^2}M\mu \leq |X|+\|w\|_\infty$. If there exists a time t_0^* such that $t_0 \leq t_0^* \leq t_0+T$, at which the solutions satisfy

$$|X(t_0^*)|+\|w(t_0^*)\|_\infty \leqslant \Omega\frac{M_2}{(1+M_0)^2}M\mu, \tag{45}$$

then they also satisfy for $t_0^* \leq t \leq t_0+T$

$$|X(t)|+\|w(t)\|_\infty \leqslant \Omega\frac{M_2}{1+M_0}M\mu, \tag{46}$$

and therefore, combining this estimate with (43) we obtain the bound (34). To see this, note that once the solutions may enter again the region where (40) holds (if this happens) then the previous analysis becomes legitimate. In particular, estimate (43) is activated, which implies that the solutions remain in the region where (46) holds. ■

We note here that the proof strategy of Lemma 2 relies on the objective to establish an ultimate boundedness property for the target system. Although, in [3], [19], a Lyapunov-like analysis is employed during the zooming-in stage, facilitating the attainment of an ultimate boundedness property (through the invariance of considered regions), our approach here relies on a small-gain ISS argument. Therefore, to establish an ultimate bound estimate, our analysis relies on a choice of the zooming in parameter Ω and the dwell-time T (where T denotes the time instant at which solutions enter a desired region), both dependent on the overshoot M_0.

We are now ready to prove Theorem 1.

Proof of Theorem 1: The inequality $|X(t_0)|+\|u(t_0)\|_\infty \leq M\overline{M}\mu$ in Lemma 1 holds with constant $\mu = \mu(t_0)$. Therefore, using (11) and the definition (20) of $\overline{M}$, the inequality (33) holds. Then applying Lemma 2 where μ is updated according to (17) the inequality (35) holds with $\mu = \mu(t) = \mu(t_0+T)$. Thus, relation (35) implies that (33) holds but with $t_0 \to t_0+T$ and $\mu = \mu(t_0+2T) = \Omega\mu(t_0+T) = \Omega\mu(t_0)$. Then by applying again Lemma 2, we have for $t_0+T < t \leq t_0+2T$ and $\mu = \mu(t_0+2T)$

$$|X(t_0+2T)|+\|w(t_0+2T)\|_\infty \leq \Omega^2\frac{M_2}{1+M_0}M\mu(t_0). \tag{47}$$

Using the estimate (34) in Lemma 2, we have for
$t_0+T < t \leq t_0+2T$

$$\begin{aligned}|X(t)|+\|w(t)\|_\infty \leqslant \max\Big\{&M_0 e^{-\delta(t-t_0-T)}\left(|X(t_0+T)|\right.\\ &\left.+\|w(t_0+T)\|_\infty\right), \Omega\frac{M_2}{1+M_0}M\mu(t)\Big\}.\end{aligned} \tag{48}$$

Therefore, since in (17) for $t_0+T < t \leq t_0+2T$, $\mu(t) = \Omega\mu(t_0)$, using (35) we obtain for $t_0+T < t \leq t_0+2T$

$$|X(t)|+\|w(t)\|_\infty \leqslant \Omega M_2 M\mu(t_0). \tag{49}$$

Repeating this procedure, we arrive at

$$|X(t)|+\|w(t)\|_\infty \leqslant \Omega^{i-1}M_2 M\mu(t_0), \tag{50}$$

for all $t_0+(i-1)T < t \leq t_0+iT$. Therefore for $t_0+(i-1)T < t \leq t_0+iT$, $i=1,2,\ldots$, we get

$$|X(t)|+\|w(t)\|_\infty \leq \Omega^{\left(\frac{t-t_0}{T}\right)}\frac{M_2 M}{\Omega}\mu(t_0). \tag{51}$$

From the definition of μ in (17) one has

$$\mu(t_0) \leq \overline{M}_1 e^{2|A|\tau}e^{2|A|t_0}\mu_0, \tag{52}$$

thus, for $t \geq t_0$ it follows from (51) that

$$|X(t)|+\|w(t)\|_\infty \leq \mu_0\overline{M}_1\frac{M_2 M}{\Omega}e^{2|A|\tau}e^{2|A|t_0}e^{(t-t_0)\frac{\ln\Omega}{T}}. \tag{53}$$

Using the method of characteristics and the variation of constants formula it is straight to prove that for $0 \leq t \leq t_0$

$$|X(t)|+\|u(t)\|_\infty \leqslant \overline{M}_1 e^{|A|t}\left(|X_0|+\|u_0\|_\infty\right). \tag{54}$$

Therefore, thanks to the inequality (11) we obtain

$$|X(t)|+\|w(t)\|_\infty \leqslant \overline{M}_1 M_1 e^{|A|t}\left(|X_0|+\|u_0\|_\infty\right), \tag{55}$$

for $0 \leqslant t \leqslant t_0$. Combining these two last estimates and the inequality (11) we get for all $t \geq 0$

$$\begin{aligned}|X(t)|+\|u(t)\|_\infty \leqslant \max\left\{e^{|A|t_0}, |X_0|+\|u_0\|_\infty\right\}\overline{M}_2\\ \times e^{|A|t_0}e^{-\frac{\ln\Omega}{T}t_0}e^{\frac{\ln\Omega}{T}t},\end{aligned} \tag{56}$$

where

$$\overline{M}_2 = \frac{\overline{M}_1}{M_2}\max\left\{\frac{M_2 M}{\Omega}e^{2|A|\tau}\mu_0, M_1\right\}. \tag{57}$$

From (31) we have

$$t_0 \leqslant \frac{1}{|A|}\ln\left[\overline{M}_3\left(|X_0|+\|u_0\|_\infty\right)\right], \tag{58}$$

with

$$\overline{M}_3 = \frac{1}{\mu_0(M\overline{M}-2\Delta)}. \tag{59}$$

Thus,

$$e^{|A|t_0} \leq \overline{M}_3\left(|X_0|+\|u_0\|_\infty\right). \tag{60}$$

Moreover,

$$-\frac{\ln\Omega}{T}t_0 \leqslant -\frac{\ln\Omega}{T}\frac{1}{|A|}\ln\left[\overline{M}_3\left(X_0+\|u_0\|_\infty\right)\right], \tag{61}$$

and thus,

$$\begin{aligned}e^{-\frac{\ln\Omega}{T}t_0} &\leqslant e^{\ln\overline{M}_3\left(|X_0|+\|u_0\|_\infty\right)^{-\frac{\ln\Omega}{T}\times\frac{1}{|A|}}}\\ &\leqslant \overline{M}_3^{-\frac{\ln\Omega}{T}\times\frac{1}{|A|}}\left(|X_0|+\|u_0\|_\infty\right)^{-\frac{\ln\Omega}{T}\times\frac{1}{|A|}}.\end{aligned} \tag{62}$$

From (60) and (62) one obtains

$$\begin{aligned}\max\left\{e^{|A|t_0}, |X_0|+\|u_0\|_\infty\right\} \leq \left(|X_0|+\|u_0\|_\infty\right)\\ \times\max\left\{\overline{M}_3, 1\right\},\end{aligned} \tag{63}$$

$$\begin{aligned}e^{|A|t_0}e^{-\frac{\ln\Omega}{T}t_0} \leq \overline{M}_3^{\left(1-\frac{\ln\Omega}{T}\times\frac{1}{|A|}\right)}\\ \times\left(|X_0|+\|u_0\|_\infty\right)^{\left(1-\frac{\ln\Omega}{T|A|}\right)}.\end{aligned} \tag{64}$$

Therefore, from (63) and (64) we arrive at

$$|X(t)| + \|u(t)\|_\infty \leq \max\left\{\overline{M}_3, 1\right\} \overline{M}_2 \overline{M}_3^{\left(1-\frac{\ln\Omega}{T}\frac{1}{|A|}\right)} \times \left(|X_0| + \|u_0\|_\infty\right)^{\left(2-\frac{\ln\Omega}{T}\frac{1}{|A|}\right)} e^{\frac{\ln\Omega}{T}t}, \tag{65}$$

which gives (29).

We now prove well-posedness. In the interval $[0, t_0]$, where there is no control, the system described by (2)–(4) with $U(t) = 0$. The existence and uniqueness of solutions within this interval are ensured by the explicit solution to the ODE subsystem (2) and the transport subsystem (3), thanks to the variation of constants formula and the characteristics method, respectively. These solutions depend only on $X_0 \in \mathbb{R}^n$ and $u_0 \in \mathcal{C}_{lpw}([0,D],\mathbb{R})$ and one has $X(t) \in AC\left([0,t_0],\mathbb{R}^n\right)$ and $u \in \mathcal{C}_{lpw}([0,D]\times[0,t_0]),\mathbb{R})$. For $t > t_0$, the system described by (2)–(4), along with the quantized controller U, defined in (15), satisfies the assumptions outlined in [16, Theorem 8.1], with $F(X,u) = AX + Bu(0)$ and $\varphi(\mu,u,X) = U(\mu,u,X)$. In particular, U defined in (15), (16) is locally Lipschitz in (X,u), given the local Lipschitzness assumption of q_1 and q_2. Therefore, the initial conditions for each interval $I_i = [t_0 + (i-1)T, t_0 + iT]$, where $i = 1, 2\ldots$, satisfy $X\left(t_0 + (i-1)T\right) \in \mathbb{R}^n$, $u\left(x, t_0 + (i-1)T\right) \in \mathcal{C}_{lpw}([0,D],\mathbb{R})$, and they are bounded due to (32) for $i = 1$ and (34) for $i \geqslant 2$, respectively. Then, the system (2)–(4) with (15), given the initial conditions $X\left(t_0 + (i-1)T\right)$ and $u(\cdot, t_0 + (i-1)T) \in \mathcal{C}_{lpw}([0,D],\mathbb{R})$, where $i = 1, 2, \cdots$, admits a unique solution such that $X(t) \in AC\left(I_{i+1},\mathbb{R}^n\right)$ and for each t, $u(\cdot,t) \in \mathcal{C}_{lpw}([0,D],\mathbb{R})$, while for each x, $u(x,\cdot) \in \mathcal{C}_{lpw}(I_{i+1},\mathbb{R})$. (This regularity of the solution is also obtained in [11], [15] in the context of transport PDE systems subject to sampling-data and quantization.) Therefore, using a proof by induction, we obtain the existence and uniqueness of a solution such that $X(t)$ is absolutely continuous in $[0,+\infty)$, while $u(\cdot,t) \in \mathcal{C}_{lpw}([0,D],\mathbb{R})$ and $u(x,\cdot) \in \mathcal{C}_{lpw}(\mathbb{R}_+,\mathbb{R})$ for each t and x, respectively.

IV. Conclusions and Current Work

Global asymptotic stability of linear systems with input delay and subject to state quantization has been established, thanks to a switched predictor-feedback control law that we introduced. The proof strategy utilized the backstepping method along with small-gain and input-to-state stability arguments. Currently, our efforts are directed towards extention to input quantization and on removing the Lipschitzness assumption on the quantizers.

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