Cruise Controllers for Vehicles on Lane-Free Ring-Roads

Dionysios Theodosis, Iasson Karafyllis and Markos Papageorgiou, *Life Fellow, IEEE*

***Abstract*— The paper introduces novel families of cruise controllers for autonomous vehicles on lane-free ring-roads. The design of the cruise controllers is based on the appropriate selection of a Control Lyapunov Function expressed on measures of the energy of the system with the kinetic energy expressed in ways similar to Newtonian or relativistic mechanics. The derived feedback laws (cruise controllers) are decentralized (per vehicle), as each vehicle determines its control input based on: (i) its own state; (ii) either only the distance from adjacent vehicles (inviscid cruise controllers) or the state of adjacent vehicles (viscous cruise controllers); and (iii) its distance from the boundaries of the ring-road.**

I. Introduction

Designing safe and efficient control strategies for autonomous vehicles constitutes a challenging topic, addressed for a variety of driving circumstances, including Adaptive Cruise Control (ACC) and Cooperative ACC (CACC) systems where vehicles can communicate with each other, see for instance [3], [4], [11], [14], [24] and references therein.

Ring-roads are of particular interest since they may give rise to phantom traffic jams when the average density is higher than the critical density, see [1], [20], [23]. The ability of connected and automated vehicles on ring-roads to dissipate traffic waves have been intensively studied and reported in both microscopic and macroscopic traffic flow models, see [2], [5], [6], [7], [8]. With the advancement of vehicular technology and the emergence of highly automated vehicles, new directions have been proposed, [13], where vehicles are not bound to traffic lanes but can freely move on the two-dimensional surface of the road and can influence the movement of adjacent vehicles all around them through sensors and communication, see [10], [12], [25].

While cruise controllers for autonomous vehicles on ring-roads and roundabouts have a long history (see for instance [4], [11], [14], [15], [17], [19], [22], [26]), they are based on a single or double-lane road and are therefore not suitable to capture the complexity of lane free ring-roads, such as the famous Charles-de-Gaulle Place roundabout in Paris, France; see [12] for a vehicle movement strategy on the latter roundabout.

The research leading to these results has received funding from the European Research Council under the European Union's Horizon 2020 Research and Innovation programme/ ERC Grant Agreement n. [833915], project TrafficFluid.

D. Theodosis and M. Papageorgiou are with the Dynamic Systems and Simulation Laboratory, Technical University of Crete, Chania, Greece. M. Papageorgiou is also with the Faculty of Maritime and Transportation, Ningbo University, Ningbo, China.I. Karafyllis is with the Dept. of Mathematics, National Technical University of Athens, Zografou Campus, 15780, Athens, Greece. (e-mail: dtheodosis@dssl.tuc.gr; iasonkar@central.ntua.gr; markos@dssl.tuc.gr)

In this paper, we extend the Control Lyapunov Function (CLF) methodologies presented in [10] to derive families of cruise controllers for autonomous vehicles that operate on lane-free ring-roads. The CLFs are based on measures of the total energy of the system and guarantee well-posedness of the closed-loop system (Lemma 1 and Lemma 2). By expressing the kinetic energy in ways similar to Newtonian or relativistic mechanics, two respective families of cruise controllers are obtained that satisfy the following properties globally (Theorem 1 and Theorem 2): (i) there are no collisions among vehicles or with the inner and outer boundaries of the ring-road; (ii) the speeds of all vehicles never exceed a given speed limit and always remain positive; (iii) the angular speeds of all vehicles converge to a given angular speed set-point; and the accelerations, the relative orientations (the deviation of the heading angle from the tangent of the circle), and rates of change of the relative orientations of all vehicles tend to zero. The proposed families of cruise controllers are decentralized (per vehicle), and each vehicle only has access to its own state, the distance from the boundaries of the ring-road and either the relative positions from adjacent vehicles (inviscid cruise controllers) or the states (relative speed, orientation, and position) of adjacent vehicles (viscous cruise controllers). A detailed analysis of the differences between the cruise controllers for the ring-road case and the straight road case can be found in [21].

The structure of the paper is as follows. Section II is devoted to the description of the employed vehicle model for the ring-road. Section III presents the families of the proposed cruise controllers and the statements of our main results. Simulation examples are included in Section IV that demonstrate the properties of the families of cruise controllers. Due to space constraints, the proofs can be found in [21].

Notation. Throughout this paper, we adopt the following notation. $\mathbb{R}_+ := [0, +\infty)$ denotes the set of non-negative real numbers. By $|x|$ we denote both the Euclidean norm of a vector $x \in \mathbb{R}^n$ and the absolute value of a scalar $x \in \mathbb{R}$. By x' we denote the transpose of a vector $x \in \mathbb{R}^n$. By $|x|_\infty = \max\{|x_i|, i = 1, \dots, n\}$ we denote the infinity norm of a vector $x = (x_1, x_2, \dots, x_n)' \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be an open set. By $C^0(A, \Omega)$, we denote the class of continuous functions on $A \subseteq \mathbb{R}^n$, which take values in $\Omega \subseteq \mathbb{R}^m$. By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \mathbb{R}^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \mathbb{R}^m$. When $\Omega = \mathbb{R}$, we write $C^0(A)$ or $C^k(A)$.

II. Model and Problem Formulation

The bicycle kinematic model ([16]) has been widely used to describe the motion of vehicles on a straight road

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= \frac{v_i}{\sigma_i} \tan(\delta_i) \\ \dot{v}_i &= F_i \end{aligned} \tag{1}$$

where (x_i, y_i), $i = 1, \ldots, n$, represent the position of the midpoint of the rear axle of vehicle i (reference point) in an inertial frame with Cartesian coordinates (X, Y), v_i is the speed of the vehicle at the point (x_i, y_i), θ_i is the heading angle (orientation) with respect to the X axis, F_i is the acceleration, δ_i is the steering angle of the front wheels relative to the orientation θ_i of the vehicle, and σ_i denotes the length of vehicle i.

To describe the motion of a vehicle on a ring-road of inner radius $R_{in} > 0$ and outer radius $R_{out} > R_{in}$ centered at $(0, 0)$, we let $r_i \in (R_{in}, R_{out})$ be the distance of the reference point of vehicle i from $(0, 0)$, and φ_i be the angular coordinate (the angle of the reference point of vehicle i from the reference direction X). Then, by using the change of coordinates $x_i = r_i \cos(\varphi_i)$, $y_i = r_i \sin(\varphi_i)$ we obtain the bicycle model in polar coordinates, as appropriate for movement on a ring-road:

$$\begin{aligned} \dot{r}_i &= -v_i \sin(s_i) \\ \dot{\varphi}_i &= \frac{v_i}{r_i} \cos(s_i) \\ \dot{s}_i &= \frac{v_i}{\sigma_i} \tan(\delta_i) - \frac{v_i}{r_i} \cos(s_i) \\ \dot{v}_i &= F_i \end{aligned} \tag{2}$$

for $i = 1, \ldots, n$, where $v_i \in (0, v_{\max})$ is the speed of the i-th vehicle at the point (r_i, ϕ_i), $v_{\max} > 0$ denotes the road speed limit, $s_i = \theta_i - \varphi_i - \frac{\pi}{2}$ is the relative orientation, i.e., the deviation of the heading angle from the tangent of the circle with radius r_i and center $(0, 0)$.

Let $\omega^* \in \left(0, \frac{v_{\max}}{R_{out}}\right)$ be given (the angular speed set point) and define the set

$$S := (R_{in}, R_{out})^n \times \mathbb{R}^n \times (-\Theta, \Theta)^n \times (0, v_{\max})^n \tag{3}$$

where $\Theta \in \left(0, \frac{\pi}{2}\right)$ is a given angle (maximum deviation of heading angle from the tangent of the ring road) that satisfies

$$\cos(\Theta) > \frac{R_{out}\omega^*}{v_{\max}} \tag{4}$$

We define the distance between vehicles by

$$\begin{aligned} &d_{i,j} := \sqrt{p_{i,j} (r_i - r_j)^2 + 2 r_i r_j (1 - \cos(\varphi_i - \varphi_j))}, \\ &\text{for } i, j = 1, \ldots, n \end{aligned} \tag{5}$$

where $p_{i,j} > 0$ are weight parameters that satisfy $p_{i,j} = p_{j,i}$ for all $i, j = 1, \ldots, n$. Notice that when $p_{i,j} = 1$ the distance metric defined by (5) coincides with the usual Euclidean distance metric (i.e., $d_{i,j}$ is equal to the usual Euclidean distance between the points (r_i, ϕ_i) and (r_j, ϕ_j) in polar coordinates). Let

$$w = (r_1, \ldots, r_n, \varphi_1, \ldots, \varphi_n, s_1, \ldots, s_n, v_1, \ldots, v_n)' \in \mathbb{R}^{4n}. \tag{6}$$

Let $L_{i,j}$, $i, j = 1, \ldots, n$, $i \neq j$, be positive constants that represent the minimum distance between vehicles i and j, with $L_{i,j} = L_{j,i}$ for $i, j = 1, \ldots, n$, $i \neq j$. Then, due to the various constraints presented above, the state space of the model (2) is

$$\Omega := \{\, w \in S \;:\; d_{i,j} > L_{i,j},\; i, j = 1, \ldots, n\,,\, j \neq i \,\} \tag{7}$$

Notice that the state-space Ω is not a linear subspace of $\mathbb{R}^{4n}$ but an open set.

Problem Statement: Design cruise controllers for vehicles operating on lane-free ring-roads that satisfy the following properties:

(P1) Well-posedness requirement: For each $w(0) \in \Omega$, there exists a unique solution $w(t) \in \Omega$ defined for all $t \geq 0$. According to (7), this requirement implies that there are no collisions between vehicles (since $d_{i,j}(t) > L_{i,j}$ for $t \geq 0$, $i, j = 1, \ldots, n$, $j \neq i$) or with the inner and outer boundaries of the ring-road (since $r_i(t) \in (R_{in}, R_{out})$ for $t \geq 0$); the speeds of all vehicles are always positive and remain below the given speed limit (since $v_i(t) \in (0, v_{\max})$ for all $t \geq 0$); and the relative orientation of each vehicle is always bounded by the given value $\Theta \in \left(0, \frac{\pi}{2}\right)$ ($s_i(t) \in (-\Theta, \Theta)$ for $t \geq 0$).

(P2) Asymptotic requirement: The relative orientation of each vehicle satisfies $\lim\limits_{t \to +\infty} (s_i(t)) = 0$ for $i = 1, ..., n$, and the angular speed of all vehicles satisfy $\lim\limits_{t \to +\infty} \left(\frac{v_i(t)}{r_i(t)}\right) = \omega^*$, $i = 1, \ldots, n$, for a given angular speed set-point $\omega^* \in \left(0, \frac{v_{\max}}{R_{out}}\right)$. Moreover, the accelerations and the rate of change of the relative orientation of all vehicles tend to zero, i.e., $\lim\limits_{t \to +\infty} (F_i(t)) = 0$, and $\lim\limits_{t \to +\infty} (\dot{s}_i(t)) = 0$, for $i = 1, ..., n$.

III. Main Results

A. *Preliminaries*

Let $V_{i,j} : (L_{i,j}, +\infty) \to \mathbb{R}_+$, $U_i : (R_{in}, R_{out}) \to \mathbb{R}_+$, $i, j = 1, \ldots, n$, $j \neq i$ be C^2 functions and $\kappa_{i,j} : (L_{i,j}, +\infty) \to \mathbb{R}_+$, $i, j = 1, \ldots, n$, $j \neq i$ be C^1 functions that satisfy the following properties

$$\lim_{d \to L_{i,j}^+} (V_{i,j}(d)) = +\infty, i, j = 1, \ldots, n, j \neq i, \tag{8}$$

$$V_{i,j}(d) = 0, \text{ for all } d \geq \lambda,\; i, j = 1, \ldots, n, j \neq i, \tag{9}$$

$$V_{i,j}(d) \equiv V_{j,i}(d),\; i, j = 1, \ldots, n, j \neq i, \tag{10}$$

$$\lim_{r \to R_{in}^+} (U_i(r)) = +\infty,\; \lim_{r \to R_{out}^-} (U_i(r)) = +\infty, i = 1, \ldots, n, \tag{11}$$

$$\kappa_{i,j}(d) \equiv \kappa_{j,i}(d), i, j = 1, \ldots, n, j \neq i, \tag{12}$$

$$\kappa_{i,j}(d) = 0, \text{ for all } d \geq \lambda,\; i, j = 1, \ldots, n, j \neq i, \tag{13}$$

where λ is a positive constant that satisfies

$$\lambda > \max\left\{L_{i,j}, i, j = 1, \ldots, n, i \neq j\right\}. \tag{14}$$

The functions $V_{i,j}$ and U_i are potential functions which have been used to avoid collisions between vehicles and road boundary violation (see [10], [24]). Condition (10) implies that if a vehicle i exerts a force to vehicle j, then vehicle j exerts the opposite force to vehicle i. In addition, properties (9) and (13) will allow to design decentralized controllers, which use real-time information (such as relative positions, relative speeds, and relative orientations) only from vehicles that are located at a distance less than $\lambda > 0$. Finally, the functions $\kappa_{i,j}$ satisfying (12), (13) will introduce a viscous-like behavior of the vehicles.

In the following sections, we will use a Control Lyapunov Function (CLF) methodology and the potential functions $V_{i,j}$, U_i in (8)-(11), to derive families of cruise controllers for autonomous vehicles on lane-free ring-roads. The construction of the Lyapunov function is based on the measures of the total energy of the system. If the kinetic energy is expressed in a fashion similar to that of Newtonian mechanics, we call the corresponding controller a *Newtonian Cruise Controller* (NCC); while, when the kinetic energy is expressed in terms similar to that of relativistic mechanics, we call the corresponding controller a *Pseudo-Relativistic Cruise Controller* (PRCC).

Finally, when $\kappa_{i,j}(d) \equiv 0$ for $i,j = 1,\dots,n, j \neq i$, we call the controller *"inviscid"* since the corresponding macroscopic model does not contain a viscosity term; otherwise, the corresponding controller is called *"viscous"*, (see [10] for the corresponding macroscopic models).

B. Newtonian Cruise Controller (NCC)

The CLF in this case is given by the formula

$$\begin{aligned} H(w) :=& \frac{1}{2}\sum_{i=1}^{n}\left(\frac{v_i}{r_i}\cos(s_i)-\omega^*\right)^2 + \frac{b}{2}\sum_{i=1}^{n} v_i^2\sin^2(s_i) \\ &+ \sum_{i=1}^{n} U_i(r_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} V_{i,j}(d_{i,j}) \\ &+ A\sum_{i=1}^{n}\left(\frac{1}{\cos(s_i)-\cos(\Theta)} - \frac{1}{1-\cos(\Theta)}\right) \end{aligned} \tag{15}$$

where $A > 0$, $b > \frac{1}{R_{in}^2}$ are parameters of the Lyapunov function. The function H in (15) is based on the total mechanical energy of the system of n vehicles. Specifically, the first two terms ($\frac{1}{2}\sum_{i=1}^{n}\left(\frac{v_i}{r_i}\cos(s_i)-\omega^*\right)^2 + \frac{b}{2}\sum_{i=1}^{n} v_i^2\sin^2(s_i)$) are related to the rotational kinetic energy of the system of n vehicles relative to an observer moving along the ring-road with angular speed equal to ω^*(as in classical mechanics). The sum of the third and fourth terms ($\sum_{i=1}^{n} U_i(r_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} V_{i,j}(d_{i,j})$), is related to the potential energy of the system. Finally, the last term of (15) ($A\sum_{i=1}^{n}\left(\frac{1}{\cos(s_i)-\cos(\Theta)} - \frac{1}{1-\cos(\Theta)}\right)$) is a penalty term that blows up when $s_i \to \pm\Theta$.

The following lemma shows that the CLF (15) has certain properties of a size function (see [18]), however, it is not a (global) size function.

Lemma 1: Let constants $R_{out} > R_{in} > 0$, $A > 0$, $v_{\max} > 0$, $\omega^* \in \left(0, \frac{v_{\max}}{R_{out}}\right)$, $L_{i,j} > 0$, $i,j = 1,\dots,n$, $i \neq j$, $\lambda > 0$ that satisfies (14)*; $\Theta \in \left(0, \frac{\pi}{2}\right)$ that satisfies* (4)*; and define the function $H : \Omega \to \mathbb{R}_+$ by means of* (15)*, where Ω is given by* (7)*. Then, there exist constants $\xi_i \in (R_{in}, R_{out})$, non-decreasing functions $\eta_i : \mathbb{R}_+ \to [\xi_i, R_{out})$, $\omega : \mathbb{R}_+ \to [0,\Theta)$, $i = 1,\dots,n$, non-increasing functions $\zeta_i : \mathbb{R}_+ \to (R_{in}, \xi_i]$, $i = 1,\dots,n$, and, for each pair $i,j = 1,\dots,n$, $i \neq j$, there exist non-increasing functions $\rho_{i,j} : \mathbb{R}_+ \to (L_{i,j}, \lambda]$ with $\rho_{i,j}(s) \equiv \rho_{j,i}(s)$, such that the following implications hold:*

$$\begin{aligned} &w \in \Omega \Rightarrow \zeta_i\left(H(w)\right) \leq r_i \leq \eta_i\left(H(w)\right), |s_i| \leq \omega\left(H(w)\right), \\ &d_{i,j} \geq \rho_{i,j}\left(H(w)\right), \text{ for } i,j = 1,\dots,n, j \neq i \end{aligned} \tag{16}$$

Let $g_k : \mathbb{R} \to \mathbb{R}$ ($k = 1,2$) be given C^1 non-decreasing functions and $f : \mathbb{R} \to \mathbb{R}_+$ be a C^1 function that satisfies

$$f(x) \geq \max(0,x), \text{ for all } x \in \mathbb{R}. \tag{17}$$

Based on the CLF (15), the NCC for (2) is given by the following equations for all $w \in \Omega$:

$$F_i = -k_i(w)\left(v_i - \frac{r_i\omega^*}{\cos(s_i)}\right) - \frac{r_i\omega^*}{\cos(s_i)}\left(\Phi_i(w) - G_i(w)\right), \tag{18}$$

$$\begin{aligned} \delta_i =& \tan^{-1}\left(\frac{\sigma_i}{r_i}\cos(s_i) - \frac{\sigma_i}{v_i a(r_i,s_i,v_i)}\left(\mu_2\sin(s_i)\right.\right. \\ &\left.\left.+ \left(bF_i\sin(s_i) + \Lambda_i(w)\right)v_i - M_i(w)\right)\right), \\ &\text{for } i = 1,\dots,n, \end{aligned} \tag{19}$$

where

$$\begin{aligned} k_i(w) =& \mu_1 + \Phi_i(w) - G_i(w) \\ &+ f\left(-\frac{v_{\max}\cos(s_i)}{v_{\max}\cos(s_i) - r_i\omega^*}\left(\Phi_i(w) - G_i(w)\right)\right), \\ &\text{for } i = 1,\dots,n \end{aligned} \tag{20}$$

and $\mu_1, \mu_2 > 0$ are positive constants. Moreover, we have

$$a(r,s,v) := \left(b - \frac{1}{r^2}\right)v^2\cos(s) + \omega^*\frac{v}{r} + \frac{A}{\left(\cos(s)-\cos(\Theta)\right)^2}, \tag{21}$$

for $(r,s,v) \in (R_{in}, R_{out}) \times (-\Theta,\Theta) \times (0, v_{\max})$

$$\begin{aligned} \Lambda_i(w) :=& \left(\frac{v_i}{r_i}\cos(s_i) - \omega^*\right)\frac{v_i}{r_i^2}\cos(s_i) - U_i'(r_i) \\ &- \sum_{j\neq i}\left(p_{i,j}\left(r_i - r_j\right) + r_j\left(1 - \cos(\varphi_i - \varphi_j)\right)\right)\frac{V_{i,j}'(d_{i,j})}{d_{i,j}} \end{aligned} \tag{22}$$

$$\Phi_i(w) := \frac{r_i}{\omega^*}\sum_{j\neq i} V_{i,j}'(d_{i,j})\frac{r_j\sin(\varphi_i - \varphi_j)}{d_{i,j}} \tag{23}$$

$$\begin{aligned} &G_i(w) := \\ &\frac{1}{\omega^*}\sum_{j\neq i}\kappa_{i,j}(d_{i,j})\left(g_1\left(\frac{v_j}{r_j}\cos(s_j)\right) - g_1\left(\frac{v_i}{r_i}\cos(s_i)\right)\right) \end{aligned} \tag{24}$$

and

$$\begin{aligned} &M_i(w) := \sum_{j\neq i}\kappa_{i,j}(d_{i,j})\left(g_2\left(\sin(s_j)\right) - g_2\left(\sin(s_i)\right)\right), \\ &\text{for } w \in \Omega,\ i = 1,\dots,n \end{aligned} \tag{25}$$

It should be noticed that when the NCC is inviscid, i.e., when either the functions $\kappa_{i,j}$ are zero or the functions g_k are constant, then, the only real-time measurement requirements of each vehicle are its own state and the distances of its adjacent vehicles (not their speeds, not their orientations). In contrast, when the NCC is viscous, i.e., when the functions $\kappa_{i,j}$ are not zero and the functions g_k are not constant, then the real-time measurement requirements of each vehicle are its own state and the states of its adjacent vehicles (including their speeds and their orientations). The term $k_i(w)$ in the acceleration $F_i(t)$ given in (18) is a state-dependent controller gain, which guarantees that the speed of each vehicle will remain positive and less than the speed limit $v_{\max}$. Finally, notice that properties (9) and (13) guarantee that the feedback laws (18) and (19), are decentralized.

C. Pseudo-Relativistic Cruise Controller (PRCC)

The CLF in this case is given by the formula

$$
\begin{aligned}
H_R(w) &:= \frac{1}{2}\sum_{i=1}^{n} \frac{\left(\frac{v_i}{r_i}\cos(s_i)-\omega^*\right)^2 + b v_i^2 \sin^2(s_i)}{(v_{\max}-v_i)v_i} \\
&+ \sum_{i=1}^{n} U_i(r_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} V_{i,j}(d_{i,j}) \\
&+ A\sum_{i=1}^{n}\left(\frac{1}{\cos(s_i)-\cos(\Theta)} - \frac{1}{1-\cos(\Theta)}\right)
\end{aligned} \tag{26}
$$

where $A > 0$, $b > \frac{1}{R_{in}^2}$ are parameters of the Lyapunov function. Notice that the kinetic energy term in H_R (i.e., the term $\frac{1}{2}\sum_{i=1}^{n} \frac{\left(\frac{v_i}{r_i}\cos(s_i)-\omega^*\right)^2 + b v_i^2\sin^2(s_i)}{(v_{\max}-v_i)v_i}$) is similar to the kinetic energy of a system of n particles in relativistic mechanics, with speed limits 0 and $v_{\max}$ in place of $-c$ and c, where c is the speed of light, which are the speed limits in relativistic mechanics. In relativistic mechanics, the kinetic energy increases to infinity when the speed of an object approaches (in absolute value) the speed of light, which indicates that no object with mass can reach the speed of light. Analogously, in (26), the kinetic energy term grows to infinity as the speed of a vehicle approaches zero or the maximum speed $v_{\max}$, thus restricting the speed of the vehicles in $(0, v_{\max})$.

The following lemma shows that the CLF (26) is a size function (see [18]) for the state space Ω defined by (7).

Lemma 2: Let constants $A > 0$, $v_{\max} > 0$, $\omega^* \in \left(0, \frac{v_{\max}}{R_{out}}\right)$, $L_{i,j} > 0$, $i,j = 1,\ldots,n$, $i \neq j$, $\lambda > 0$ *that satisfies* (14), $\Theta \in \left(0, \frac{\pi}{2}\right)$ *that satisfies* (5), *and define the function* $H_R : \Omega \to \mathbb{R}_+$ *by means of* (26), *where* Ω *is given by* (7). *Then, there exist constants* $\bar{v} \in (0, v_{\max})$, $\xi_i \in (R_{in}, R_{out})$, $i = 1,\ldots,n$, *non-decreasing functions* $\eta_i : \mathbb{R}_+ \to [\xi_i, R_{out})$, $i = 1,\ldots,n$, $\psi : \mathbb{R}_+ \to [\bar{v}, v_{\max})$, $\omega : \mathbb{R}_+ \to [0,\Theta)$, *non-increasing functions* $\zeta_i : \mathbb{R}_+ \to (R_{in}, \xi_i]$,$i = 1,\ldots,n$, $\chi : \mathbb{R}_+ \to (0, \bar{v}]$, *and, for each pair* $i,j = 1,\ldots,n$, $i \neq j$, *there exist non-increasing functions* $\rho_{i,j} : \mathbb{R}_+ \to (L_{i,j}, \lambda]$ *with* $\rho_{i,j}(s) \equiv \rho_{j,i}(s)$, *such that the following implications hold:*

$$
\begin{aligned}
& w \in \Omega \Rightarrow \\
& \chi\left(H_R(w)\right) \leq v_i \leq \psi\left(H_R(w)\right), |s_i| \leq \omega\left(H_R(w)\right), \\
& \zeta_i\left(H_R(w)\right) \leq r_i \leq \eta_i\left(H_R(w)\right), d_{i,j} \geq \rho_{i,j}\left(H_R(w)\right) \\
& \text{for } i,j = 1,\ldots,n,\ j \neq i
\end{aligned} \tag{27}
$$

Let $g_k : \mathbb{R} \to \mathbb{R}$ $(k = 1,2)$ be given C^1 non-decreasing functions and let also $f_j : \mathbb{R} \to \mathbb{R}$,$j = 1,2$, be C^1 functions that satisfy:

$$
f_j(0) = 0 \text{ and } x\, f_j(x) > 0, \text{ for } x \neq 0,\ j = 1,2. \tag{28}
$$

The Pseudo-Relativistic Cruise Controller (PRCC) for (2) that correspond to the CLF (26) is given by the following equations for all $w \in \Omega$:

$$
\begin{aligned}
F_i = -\frac{1}{q(r_i,s_i,v_i)} &\left(f_1\left(\frac{v_i}{r_i}\cos(s_i)-\omega^*\right)\right. \\
&\left.+\,\omega^*\left(\Phi_i(w)-G_i(w)\right)\right), \text{ for } i = 1,\ldots,n,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\delta_i = \tan^{-1}&\left(\frac{\sigma_i}{r_i}\cos(s_i) - \frac{\sigma_i}{\gamma(r_i,s_i,v_i)v_i}\left(f_2(\sin(s_i))\right.\right. \\
&\left.\left.-\left(\zeta_i(s_i,v_i)F_i + Z_i(w)\right)v_i - M_i(w)\right)\right), \\
&\text{for } i = 1,\ldots,n
\end{aligned} \tag{30}
$$

where Φ_i, G_i and M_i are given by (23), (24) and (25), respectively, and

$$
q(r,s,v) = \frac{v_{\max} v\cos(s) - 2rv\omega^* + r\omega^* v_{\max}}{2r(v_{\max}-v)^2 v^2} \tag{31}
$$

$$
\begin{aligned}
\gamma(r,s,v) = &\frac{A}{(\cos(s)-\cos(\Theta))^2} + \frac{v\cos(s)}{v_{\max}-v}\left(b - \frac{1}{r^2}\right) \\
&+ \frac{\omega^*}{r(v_{\max}-v)}
\end{aligned} \tag{32}
$$

$$
\zeta(s,v) = \frac{b v_{\max}\sin(s)}{2(v_{\max}-v)^2 v} \tag{33}
$$

for $(r,s,v) \in (R_{in}, R_{out}) \times (-\Theta,\Theta) \times (0, v_{\max})$, $b > \frac{1}{R_{in}^2}$ and

$$
\begin{aligned}
Z_i(w) := &\left(\frac{v_i}{r_i}\cos(s_i)-\omega^*\right)\frac{\cos(s_i)}{(v_{\max}-v_i)r_i^2} - U_i'(r_i) \\
&- \sum_{j\neq i}\left(p_{i,j}\left(r_i - r_j\right) + r_j\left(1-\cos(\varphi_i-\varphi_j)\right)\right)\frac{V_{i,j}'(d_{i,j})}{d_{i,j}}
\end{aligned} \tag{34}
$$

for $w \in \Omega$. Notice that the assumption $b > \frac{1}{R_{in}^2}$ guarantees that $\gamma(r,s,v) > 0$ for all $(r,s,v) \in (R_{in}, R_{out}) \times (-\Theta,\Theta) \times (0, v_{\max})$.

The pseudo-relativistic feedback laws (29) and (30) are derived by using the CLF (26) which is also a size function.Finally, notice that properties (9) and (13) guarantee that the feedback laws (29) and (30), are decentralized (per vehicle) and depend only on adjacent vehicles' states, namely vehicles that are located at a distance less than $\lambda > 0$.

D. Statements of Main Results

The following Theorems guarantee that the closed-loop system (2) with (18), (19) and (2) with (29) and (30) satisfy properties (P1) and (P2).

Theorem 1: For every $w_0 \in \Omega$ *the initial-value problem* (2) *with* (18), (19), (20) *and* $w(0) = w_0$ *has a unique solution* $w(t)$*, defined for all* $t \geq 0$*, that satisfies* $w(t) \in \Omega$ *for all* $t \geq 0$*, as well as*

$$\lim_{t\to+\infty}(s_i(t)) = 0, \ \lim_{t\to+\infty}\left(\frac{v_i(t)}{r_i(t)}\right) = \omega^*, \text{ for } i = 1, \dots, n, \tag{35}$$

$$\lim_{t\to+\infty}(\dot{s}_i(t)) = 0, , \ \lim_{t\to+\infty}(F_i(t)) = 0 \text{ for } i = 1, \dots, n, \tag{36}$$

Furthermore, there exists a non-decreasing function $P_1 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $|F_i(t)| + |\delta_i(t)| \leq P_1(H(w_0))$ for $t \geq 0, \ i = 1, \dots, n$

Theorem 2: For every $w_0 \in \Omega$ *the initial-value problem* (2) *with* (29), (30) *and* $w(0) = w_0$ *has a unique solution* $w(t)$*, defined for all* $t \geq 0$*, that satisfies* $w(t) \in \Omega$ *for all* $t \geq 0$*, as well as* (35) *and* (36)*. Furthermore, there exists a non-decreasing function* $P_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $|F_i(t)| + |\delta_i(t)| \leq P_2(H_R(w_0))$ for $t \geq 0, \ i = 1, \dots, n$

IV. Numerical Simulations

In the simulation results below, we apply the proposed decentralized cruise controllers (NCC and PRCC) for both the viscous and inviscid case. Specifically, we consider a group of $n = 10$ vehicles on a lane-free ring-road with $R_{in} = 20$ and $R_{out} = 60$. The vehicle-repulsive potential functions $V_{i,j}$ and the boundary-repulsive potential function U_i for both the NCC and PRCC are specified as

$$V_{i,j} = \begin{cases} q_1 \frac{(\lambda-d)^3}{d-L_{i,j}} & , L_{i,j} < d \leq \lambda \\ 0 & , d > \lambda \end{cases} \tag{37}$$

$$U_i(r) = \begin{cases} 0 & , |r - R_m| \leq c \\ \frac{(r-R_m-c)^3(r-R_m+c)^3}{(r-R_{in})(R_{out}-r)} & , |r - R_m| > c \end{cases} \tag{38}$$

where $0 < c < \frac{R_{out}-R_{in}}{2}$, $q_1 > 0$ are design parameters, $R_m = \frac{R_{in}+R_{out}}{2}$, and $L_{i,j} = L$, $p_{i,j} = p$, $i, j = 1, \dots, n$, $j \neq i$. Notice that $V_{i,j}$ and U_i in (37) and (38) satisfy properties (8), (9), (10) and (11), (12), respectively. For small values of q_1, the values of $V_{i,j}$ (and consequently the acceleration F_i) will be smaller away from L, but will increase more sharply as d approaches L. By adjusting the value of c, we can create an annulus $A = \{r \in (R_{in}, R_{out}) : R_m - c \leq r \leq R_m + c\}$ in the ring-road that satisfies $U_i(r) = 0$, $r \in A$, which may affect the final configuration of the vehicles relative to the boundaries R_{in} and R_{out}. Notice that $V_{i,j}$ and U_i above, satisfy (8), (9), (10), and (11), respectively. Finally, for both the NCC and PRCC we consider that the viscosity is given by

$$g_1(x) = g_2(x) = x$$
$$\kappa_{i,j}(d) = \begin{cases} q_2(\lambda - d)^2 & L < d \leq \lambda \\ 0 & d > \lambda \end{cases} \tag{39}$$

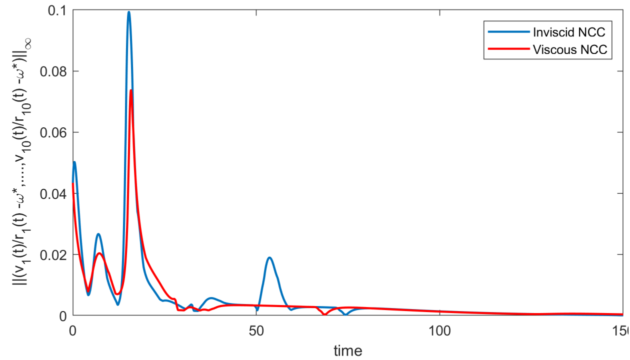

Fig. 1. Evolution and convergence of $\left\| \frac{v_1(t)}{r_1(t)} - \omega^*, \dots, \frac{v_{10}(t)}{r_{10}(t)} - \omega^* \right\|_\infty$ for the Inviscid and Viscous NCC.

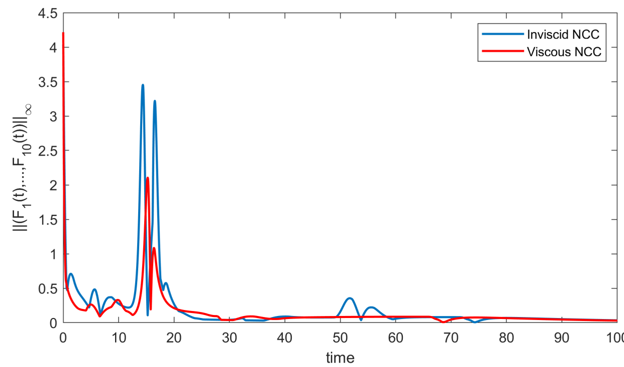

Fig. 2. Evolution and convergence of $\|(F_1(t), ..., F_{10}(t))\|_\infty$ for the Inviscid and Viscous NCC.

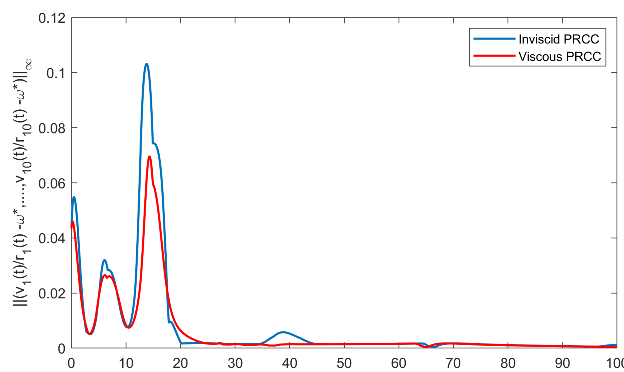

Fig. 3. Evolution and convergence of $\left\| \frac{v_1(t)}{r_1(t)} - \omega^*, \dots, \frac{v_{10}(t)}{r_{10}(t)} - \omega^* \right\|_\infty$ for the Inviscid and Viscous PRCC.

where $q_2 \geq 0$ is a design parameter which can be selected to increase or decrease the effects of viscosity. Notice that if $q_2 \equiv 0$, then we obtain the inviscid NCC and PRCC cruise controllers. Moreover, we select for the NCC

$$f(x) = \frac{1}{2\varepsilon}\begin{cases} 0 & x \leq -\varepsilon \\ (x+\varepsilon)^2 & -\varepsilon < x < 0 \\ \varepsilon^2 + 2\varepsilon x & x \geq 0 \end{cases} \tag{40}$$

that satisfies (17), where $\varepsilon > 0$ is design parameter. Finally, for the PRCC we select $f_1(x) = \mu_1 x$ and $f_2(x) = \mu_2 x$, $x \in \mathbb{R}$, where $\mu_1, \mu_2 > 0$.

To verify numerically and illustrate the results of Theorem

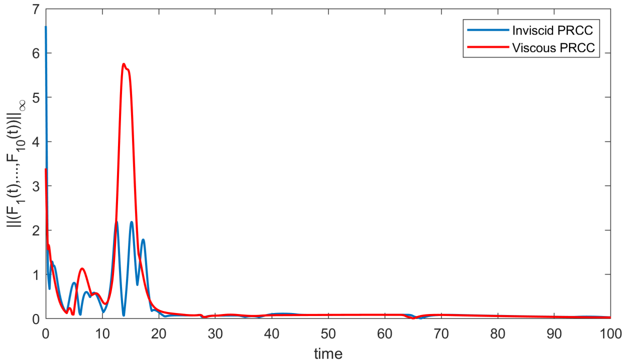

Fig. 4. Evolution and convergence of $\|(F_1(t), ..., F_{10}(t))\|_\infty$ for the Inviscid and Viscous PRCC.

1, and Theorem 2, we assume that all vehicles have length $\sigma = 5$ and we set the angular speed set-point $\omega^* = 0.15$, the maximum speed $v_{\max} = 10$ and select $\Theta = 0.17$ in order to satisfy condition (4) (recall that $R_{out} = 60$). Finally, we set $\varepsilon = 0.2$, $\mu_1 = 0.3$, $\mu_2 = 10^2$, $p_{i,j} = 5.11$, for all $i, j = 1, \dots, n$, $q_1 = 3 * 10^{-3}$, $\lambda = 20$, $A = 0.5$, $b = 1$, $c = 10$, $q_2 = 0.1$ (for the viscous case) and $L = 6$.

Figure 1 shows the convergence of $\frac{v_i(t)}{r_i(t)}$ to ω^* for both the inviscid and viscous NCC. It is seen that by adding viscosity to the system, we have smoother convergence to ω^*. This is also illustrated in Figure 2 which shows the evolution of the acceleration $||F_1(t), ..., F_{10}(t)||_\infty$.

For the PRCC we select $q_1 = 3 \cdot 10^{-5}$ and $q_2 = 0.1$ for the viscous case. Fig. 3 shows the evolution and convergence of $\frac{v_i(t)}{r_i(t)}$ to ω^* for the Inviscid and Viscous PRCC. Fig. 4 shows the evolution of $||F_1(t), ..., F_{10}(t)||_\infty$ for the viscous and inviscid PRCC.

V. Conclusions

The paper introduced two families of cruise controllers for autonomous vehicles operating on lane-free ring-roads. By expressing the Control Lyapunov Functions on measures of the energy of the system with the kinetic energy expressed in ways similar to Newtonian or relativistic mechanics, we derived decentralized feedback laws (cruise controllers) that guarantee collision avoidance between vehicles and with the boundary of the ring-road; that the speeds of all vehicles are always positive and remain below a given speed limit; and that all angular speeds converge to a given speed set-point.

References

[1] Bando, M., Hesebem K., A. Nakayama, A. Shibata, and Y. Sugiyama. "Dynamical Model of Traffic Congestion and Numerical Simulation", *Physical Review E*, 51, 1995, 1035–1042.

[2] Cui S., B. Seibold, R. Stern and D. B. Work, "Stabilizing Traffic Flow via a Single Autonomous Vehicle: Possibilities and Limitations," *2017 IEEE Intelligent Vehicles Symposium (IV)*, 2017, 1336–1341.

[3] Diakaki, C., M. Papageorgiou, I. Papamichail, I. Nikolos, "Overview and Analysis of Vehicle Automation and Communication Systems from a Motorway Traffic Management Perspective", *Transportation Research Part A: Policy and Practice,* 75, 2015, 147–165.

[4] Debada, E., and D. Gillet, "Merging into Single-Lane Roundabouts in the Presence of Uncertainty", *IEEE 21st International Conference on Intelligent Transportation Systems (ITSC)*, 2019, 3168–3175.

[5] Giammarino V., S. Baldi, P. Frasca and M. L. D. Monache, "Traffic Flow on a Ring with a Single Autonomous Vehicle: An Interconnected Stability Perspective," *IEEE Transactions on Intelligent Transportation Systems*, 22, 2021, 4998–5008.

[6] Hayat, A., B. Piccoli, and S. Truong. "Dissipation of Traffic Jams Using a Single Autonomous Vehicle on a Ring Road", 2021, (see hal-03354282).

[7] Huang K., Q. Du, "Stability of a Nonlocal Traffic Flow Model for Connected Vehicles", *SIAM Journal on Applied Mathematics, 82,* 2022, 221–243.

[8] Karafyllis, I., D. Theodosis and M. Papageorgiou, "Analysis and Control of a Non-Local PDE Traffic Flow Model", *International Journal of Control,* 95, 2022, 660–678.

[9] Karafyllis, I. , D. Theodosis and M. Papageorgiou, "Stability Analysis of Nonlinear Inviscid Microscopic and Macroscopic Traffic Flow Models of Bidirectional Cruise-Controlled Vehicles", *IMA Journal of Mathematical Control and Information*, 39 , 2022, 609–642.

[10] Karafyllis, I., D. Theodosis and M. Papageorgiou, "Constructing Artificial Traffic Fluids by Designing Cruise Controllers", *Systems & Control Letters*, 167, 2022, 105317.

[11] Marshall, J. A., M. E. Broucke and B. A. Francis, "Formations of Vehicles in Cyclic Pursuit," *IEEE Transactions on Automatic Control*, 49, 2004, 1963–1974.

[12] Naderi, M., M. Papageorgiou, I. Karafyllis, I. Papamichail, "Automated Vehicle Driving on Large Lane-free Roundabouts". Accepted in *25th IEEE International Conference on Intelligent Transportation Systems (ITSC)*, Macau, China, 2022.

[13] Papageorgiou, M., K.-S. Mountakis, I. Karafyllis, I. Papamichail, and Y. Wang, "Lane-Free Artificial-Fluid Concept for Vehicular Traffic", *Proceedings of the IEEE,* 109, 2021, 114–121.

[14] Peters A.A., R. H. Middleton, and O. Mason, "Cyclic Interconnection for Formation Control of 1-D Vehicle Strings*", European Journal of Control*, 27, 2016, 36–44.

[15] Pooladsanj M., K. Savla and P. Ioannou, "Vehicle Following On a Ring Road Under Safety Constraints: Role of Connectivity and Coordination," in *IEEE Transactions on Intelligent Vehicles*, doi: 10.1109/TIV.2022.3149972.

[16] Rajamani, R., *Vehicle Dynamics and Control*, Springer-Verlag, New York, 2012.

[17] Rogge J. A., and D. Aeyels, "Vehicle Platoons Through Ring Coupling", *IEEE Transactions on Automatic Control*, 53, 2008, 1370–1377.

[18] Sontag E. D., "Remarks on Input to State Stability of Perturbed Gradient Flows, Motivated by Model-free Feedback Control Learning", *Systems & Control Letters*, 161, 2022, 105138.

[19] Stüdli S., M. M. Seron, R. H. Middleton, "Vehicular Platoons in Cyclic Interconnections,", *Automatica*, 94, 2018, 283–293.

[20] Sugiyama, Y., Fukui, M., Kikuchi, M., Hasebe, K., Nakayama, A., Nishinari, K., Yukawa, S.: "Traffic Jams Without Bottlenecks – Experimental Evidence for the Physical Mechanism of the Formation of a Jam", *New Journal of Physics*, 10, 2008, 033001.

[21] Theodosis D., Karafyllis, I., and M. Papageorgiou, "Cruise Controllers for Lane-Free Ring-Roads based on Control Lyapunov Functions ", see arXiv:2208.01977 [math.OC], 2022.

[22] Tian, R., S. Li, N. Li, I. Kolmanovsky, A. Girard, and Y. Yildiz, "Adaptive Game-Theoretic Decision Making for Autonomous Vehicle Control at Roundabouts", *IEEE Conference on Decision and Control (CDC)*, 2018, 321–326.

[23] Wilson, R.E., "Mechanisms for Spatio-Temporal Pattern Formation in Highway Traffic Models", *Philosophical Transactions of the Royal Society A*, 366, 2008, 2017–2032.

[24] Wolf M. T. and J. W. Burdick, "Artificial Potential Functions for Highway Driving With Collision Avoidance," *2008 IEEE International Conference on Robotics and Automation*, Pasadena, CA, USA, 2008, 3731–3736.

[25] Yanumula V.K., P. Typaldos, D. Troullinos, M. Malekzadeh, I. Papamichail, and M. Papageorgiou, "Optimal Path Planning for Connected and Automated Vehicles in Lane-Free Traffic", *24^{th} IEEE International Conference on Intelligent Transportation (ITSC 2021)*, Indianapolis, USA, 2021, 3545–3552.

[26] Zhang, J., X. Ma, and Y. Wang, "Study on Stability and Scalability of Vehicular Platoon with a Bidirectional Ring Interconnection". *IET Intelligent Transportation Systems,* 15, 2021, 1388–1400.