Artificial Traffic Fluids Emerging from the Design of Cruise Controllers

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***Abstract*— In this paper, we present two Control Lyapunov Function based families of cruise controllers for the two-dimensional movement of autonomous vehicles on lane-free roads using the bicycle kinematic model. The control Lyapunov functions are based on measures of the energy of the system with the kinetic energy expressed in ways similar to Newtonian or relativistic mechanics. The derived feedback laws (cruise controllers) are decentralized, as each vehicle determines its control input based on its own speed and on the relative speeds and distances from adjacent vehicles and from the boundary of the road. Moreover, the corresponding macroscopic models are derived, obtaining fluid-like models that consist of a conservation equation and a momentum equation with pressure and viscous terms. Finally, we show that, by selecting appropriately the parameters of the feedback laws, we get free hand to create an artificial fluid that approximates the emerging traffic flow.**

I. Introduction

The mathematical description of conventional traffic through microscopic models has been studied extensively with various contributions and applications. Recent advances in technology have revolutionized vehicle automation with different kinds of driver support systems (see for instance [9], [16], [18]). In the era of connected and automated vehicles, new perspectives and principles have also been suggested [17], where autonomous vehicles can move on the two-dimensional surface of lane-free roads ([11], [23]) without abiding to lane discipline, something that may improve traffic flow and increase capacity of highways and arterials.

To study the collective behavior of vehicles through aggregate variables (flow, density, and mean speed of vehicles), macroscopic traffic flow modelling was suggested in the 1950s and continued to this day with a variety of models and approaches, see for instance [3], [5], [7], [10], [13], [24], [25] and references therein. Furthermore, several methodologies have been suggested to derive macroscopic models from microscopic models, see for instance [3], [4], [7], [8], [21] and references therein.

In this paper, we present a Control Lyapunov Function (CLF) methodology to derive families of cruise controllers for autonomous vehicles on lane-free roads. The construction of the CLF is based on measures of the total energy of the system. By expressing the kinetic energy in ways similar to Newtonian or relativistic mechanics, two respective families of cruise controllers are obtained that satisfy the following properties globally (see Theorem 1 and Theorem 2): (i) there are no collisions among vehicles or with the boundary of the road; (ii) the speeds of all vehicles are always positive and remain below a given speed limit; (iii) the speeds of all vehicles converge to a given speed set-point; and (iv) the accelerations, lateral speeds, and orientations of all vehicles tend to zero. The proposed families of cruise controllers are decentralized (per vehicle) and require either the measurement only of the distances from adjacent vehicles (inviscid cruise controllers) or the measurement of speeds of and distances from adjacent vehicles (viscous cruise controllers). Theorem 1 extends the corresponding result presented in [11], by introducing a viscous term on the controller and by considering vehicles of different sizes.

Finally, using the methodology in [7], we formally derive the macroscopic models that correspond to the closed-loop systems with the derived cruise controllers (Section IV). The resulting macroscopic models are very similar to models describing the flow of a Newtonian, compressible fluid in a porous or non-porous medium. We also provide the explicit formulae that relate the physical characteristics of the "traffic fluid" to the parameters of the cruise controllers. This implies that, by changing the functions and the parameters of the cruise controllers, we can actually determine the physical characteristics of the "traffic fluid".

Due to space constraints, the proofs of all results and the derivation of the macroscopic models are omitted and can be found in [12].

Notation. Throughout this paper, we adopt the following notation. By $\mathbb{R}_+ := [0,+\infty)$ we denote the set of non-negative real numbers. By $|x|$ we denote both the Euclidean norm of a vector $x \in \mathbb{R}^n$ and the absolute value of a scalar $x \in \mathbb{R}$. By x' we denote the transpose of a vector $x \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be an open set. By $C^0(A)$, we denote the class of continuous functions on $A \subseteq \mathbb{R}^n$, which take values in $\mathbb{R}^m$. By $C^k(A)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \mathbb{R}^n$ with continuous derivatives of order k, which take values in $\mathbb{R}^m$.

II. Vehicular Model Description

Consider n vehicles on a lane-free road of constant width $2a > 0$, where the movement of each vehicle $i \in \{1,...,n\}$ is described by the model

$$\dot{x}_i = v_i \cos(\theta_i),\ \dot{y}_i = v_i \sin(\theta_i),\ \dot{\theta}_i = \sigma_i^{-1} v_i \tan(\delta_i),\ \dot{v}_i = F_i \tag{1}$$

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where $\sigma_i > 0$ is the length of vehicle i (a constant). Here, $(x_i, y_i) \in \mathbb{R} \times (-a, a)$ is the reference point of the i-th vehicle in an inertial frame with Cartesian coordinates (X, Y), with $i \in \{1, ..., n\}$ and is placed at the midpoint of the rear axle of the vehicle; $v_i \in (0, v_{\max})$ is the speed of the i-th vehicle at the point (x_i, y_i), where $v_{\max} > 0$ denotes the road speed limit; $\theta_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the heading angle (orientation) of the i-th vehicle with respect to the X axis; δ_i is the steering angle of the front wheels relative to the orientation θ_i of the i-th vehicle; and F_i is the acceleration of the i-th vehicle. Model (1) is known as the bicycle kinematic model (see [18]), and has been used to represent vehicles due to its simplicity to capture vehicle motion. In what follows, we use the input transformation $\delta_i := \arctan\left(\sigma_i v_i^{-1} u_i\right)$, to obtain the following system

$$\dot{x}_i = v_i \cos(\theta_i),\ \ \dot{y}_i = v_i \sin(\theta_i),\ \ \dot{\theta}_i = u_i,\ \ \dot{v}_i = F_i \tag{2}$$

for $i = 1, ..., n$, where u_i, F_i, are the inputs of the system.

Let $v^* \in (0, v_{\max})$ be a given speed set-point and define the set

$$S := \mathbb{R}^n \times (-a, a)^n \times (-\varphi, \varphi)^n \times (0, v_{\max})^n \tag{3}$$

where $\varphi \in \left(0, \frac{\pi}{2}\right)$ is an angle that satisfies

$$\cos(\varphi) > v^*/v_{\max}. \tag{4}$$

The set S in (3) describes all possible states of the system of n vehicles. Specifically, each vehicle should stay within the road, i.e., $(x_i, y_i) \in \mathbb{R} \times (-a, a)$ for $i = 1, ..., n$; moreover, the vehicles should not be able to turn perpendicular to the road, hence it should hold that $\theta_i \in (-\varphi, \varphi)$ for $i = 1, ..., n$. The constant φ can be understood as a safety constraint, which restricts the movement of a vehicle; finally, the speeds of all vehicles should always be positive, i.e., no vehicle moves backwards; and all vehicles should respect the road speed limit.

We define the distance between vehicles by

$$d_{i,j} = \sqrt{(x_i - x_j)^2 + p_{i,j}(y_i - y_j)^2},\ \text{ for } i, j = 1, ..., n \tag{5}$$

where $p_{i,j} \geq 1$ are weighting factors that satisfy $p_{i,j} = p_{j,i}$ for all $i, j = 1, ..., n$. Note that, for $p_{i,j} = 1$, we obtain the standard Euclidean distance, while for larger values of $p_{i,j} > 1$, we have an "elliptical" metric, which can approximate more accurately the dimensions of a vehicle. For the case of n vehicles of equal length, the optimal selection of a single p can be found in [11]. Let

$$w = (x_1, ..., x_n, y_1, ..., y_n, \theta_1, ..., \theta_n, v_1, ..., v_n)' \in \mathbb{R}^{4n} \tag{6}$$

Due to the various constraints explained above, the state space of the model (2) is the set

$$\Omega := \{\, w \in S \ : \ d_{i,j} > L_{i,j},\ i, j = 1, ..., n\,,\, j \neq i \,\} \tag{7}$$

where $L_{i,j}$, $i, j = 1, ..., n$, $i \neq j$, are positive constants and represent the minimum distance between a vehicle i and a vehicle j, with $L_{i,j} = L_{j,i}$ for $i, j = 1, ..., n$, $i \neq j$. To have a well-posed closed-loop system on the state space $\Omega \subset \mathbb{R}^{4n}$, the control inputs u_i and F_i, $i = 1, ..., n$, should be given by appropriate feedback laws which are designed in such a way that every solution of (2) satisfies the following implication: $w(0) \in \Omega \Rightarrow w(t) \in \Omega$ for all $t \geq 0$ (see Section III).

Finally, it should be noted that model (2) with state space given by (7) is an extension of the model given in [11], where all vehicles were assumed to be identical and all distances between vehicles were given by the same value of p in (5).

III. Two Families of Cruise Controllers

A. Preliminaries

Our main objective is to design cruise controllers for vehicles operating on lane-free roads that satisfy the following properties:

(P1) Well-posedness requirement: For each $w(0) \in \Omega$, there exists a unique solution $w(t) \in \Omega$ defined for all $t \geq 0$. According to (7), this requirement implies that there are no collisions between vehicles (since $d_{i,j}(t) > L_{i,j}$ for $t \geq 0$, $i, j = 1, ..., n$, $j \neq i$) or with the boundary of the road (since $y_i(t) \in (-a, a)$ for $t \geq 0$); the speeds of all vehicles are always positive and remain below the given speed limit (since $v_i(t) \in (0, v_{\max})$ for all $t \geq 0$); and the orientation of each vehicle is always bounded by the given value $\varphi \in \left(0, \frac{\pi}{2}\right)$ (since $\theta_i(t) \in (-\varphi, \varphi)$ for $t \geq 0$).

(P2) Asymptotic requirement: The orientation of each vehicle satisfies $\lim\limits_{t \to +\infty} (\theta_i(t)) = 0$ for $i = 1, ..., n$, and the speeds of all vehicles satisfy $\lim\limits_{t \to +\infty} (v_i(t)) = v^*$, $i = 1, ..., n$, for a given a longitudinal speed set-point $v^* \in (0, v_{\max})$. Moreover, the accelerations, angular speeds, and lateral speeds of all vehicles tend to zero, i.e., $\lim\limits_{t \to +\infty} (F_i(t)) = 0$, $\lim\limits_{t \to +\infty} (u_i(t)) = 0$, and $\lim\limits_{t \to +\infty} (\dot{y}_i(t)) = 0$, for $i = 1, ..., n$.

Let $V_{i,j} : (L_{i,j}, +\infty) \to \mathbb{R}_+$, $U_i : (-a, a) \to \mathbb{R}_+$, $i, j = 1, ..., n$, $j \neq i$ be C^2 functions and let $\kappa_{i,j} : (L_{i,j}, +\infty) \to \mathbb{R}_+, i, j = 1, ..., n$, $j \neq i$ be C^1 functions that satisfy the following properties

$$\lim_{d \to L_{i,j}^+} (V_{i,j}(d)) = +\infty \tag{8}$$

$$V_{i,j}(d) = 0,\ \text{ for all } d \geq \lambda \tag{9}$$

$$V_{i,j}(d) \equiv V_{j,i}(d),\ i, j = 1, ..., n, j \neq i \tag{10}$$

$$\lim_{y \to (-a)^+} (U_i(y)) = +\infty,\ \lim_{y \to a^-} (U_i(y)) = +\infty \tag{11}$$

$$U_i(0) = 0 \tag{12}$$

$$\kappa_{i,j}(d) = 0,\ \text{ for all } d \geq \lambda \tag{13}$$

$$\kappa_{i,j}(d) \equiv \kappa_{j,i}(d),\ i, j = 1, ..., n, j \neq i \tag{14}$$

where λ is a positive constant that satisfies

$$\lambda > \max\{L_{i,j}, i, j = 1, ..., n, i \neq j\}. \tag{15}$$

The families of functions $V_{i,j}$ and U_ι in (8), (9), (10), and (11), (12), respectively, are potential functions, which have been used to avoid collisions between vehicles and road boundary violation (see for instance [22]). Condition (10) implies that if a vehicle i exerts a force to vehicle j, then vehicle j exerts the opposite force to vehicle i. The functions $\kappa_{i,j}$ are used in the subsequent analysis for the introduction of a viscous-like behavior of the vehicles.

We exploit next a Control Lyapunov Function (CLF) methodology and the potential functions above to derive families of cruise controllers for autonomous vehicles on lane-free roads that satisfy properties (P1) and (P2). The construction of the CLFs is based on measures of the total energy of the system. Depending on how the kinetic energy of the system is expressed, we obtain two different families of cruise controllers. If the kinetic energy is expressed in a fashion similar to that of Newtonian mechanics, we call the corresponding controller a *Newtonian Cruise Controller* (NCC); while, when the kinetic energy is expressed in terms similar to those of relativistic mechanics, we call the corresponding controller a *Pseudo-Relativistic Cruise Controller* (PRCC). Finally, when $\kappa_{i,j}(d) \equiv 0$ for $i, j = 1, ..., n, j \neq i$, we call the controller *"inviscid"* since the corresponding macroscopic model does not contain a viscosity term; otherwise, the corresponding controller is called *"viscous"*.

B. Newtonian Cruise Controller (NCC)

The CLF in this case is given by the formula

$$\begin{aligned} H(w) :=& \frac{1}{2}\sum_{i=1}^{n}(v_i\cos(\theta_i)-v^*)^2+\frac{b}{2}\sum_{i=1}^{n}v_i^2\sin^2(\theta_i) \\ &+\sum_{i=1}^{n}U_i(y_i)+\frac{1}{2}\sum_{i=1}^{n}\sum_{i\neq j}V_{i,j}(d_{i,j}) \\ &+A\sum_{i=1}^{n}\left(\frac{1}{cos(\theta_i)-\cos(\varphi)}-\frac{1}{1-\cos\varphi}\right) \end{aligned} \tag{16}$$

where $A > 0$, $b > 1 - \frac{v^*}{v_{\max}} > 0$ (recall that $v^* \in (0, v_{\max})$) are parameters of the controller and the Lyapunov function. The function H in (16) is based on the total mechanical energy of the system of n vehicles. Specifically, the first two terms ($\frac{1}{2}\sum_{i=1}^{n}(v_i\cos(\theta_i)-v^*)^2+\frac{b}{2}\sum_{i=1}^{n}v_i^2\sin^2(\theta_i)$) are related to the kinetic energy of the system of n vehicles relative to an observer moving along the $x-$direction with speed equal to v^*(as in classical mechanics); they penalize the deviation of the longitudinal and lateral speeds from their desired values v^* and zero, respectively. The sum of the third and fourth term ($\sum_{i=1}^{n}U_i(y_i)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}V_{i,j}(d_{i,j})$), which are based on the potential functions (8) and (11), is related to the potential energy of the system. Finally, the last term of (16) is a penalty term that blows up when $\theta_i \to \pm\varphi$.

While the CLF (16) has characteristics of a size function (see [20]), it is not a (global) size function, since H takes finite values for $v_i \notin (0, v_{\max})$. The following lemma shows certain properties of the Lyapunov function H.

Lemma 1: *Let constants $A > 0$, $v_{\max} > 0$, $v^* \in (0, v_{\max})$, $L_{i,j} > 0$, $i, j = 1, ..., n$, $i \neq j$, $\lambda > 0$ that satisfies* (15)*, $\varphi \in \left(0, \frac{\pi}{2}\right)$ that satisfies* (4)*, and define the function $H : \Omega \to \mathbb{R}_+$ by means of* (16)*, where Ω is given by* (7)*. Then, there exist non-decreasing functions $\omega : \mathbb{R}_+ \to [0, \varphi)$, $\eta_i : \mathbb{R}_+ \to [0, a)$, $i = 1, ..., n$, and for each pair $i, j \in \{1, ..., n\}$, $i \neq j$, there exist non-increasing functions $\rho_{i,j} : \mathbb{R}_+ \to (L_{i,j}, \lambda]$ with $\rho_{i,j}(s) \equiv \rho_{j,i}(s)$, such that the following implications holds:*

$$\begin{aligned} w \in& \Omega \Rightarrow |\theta_i| \leq \omega\left(H(w)\right), |y_i| \leq \eta_i\left(H(w)\right), \\ &d_{i,j} \geq \rho_{i,j}\left(H(w)\right), \text{ for } i, j = 1, ..., n\,,\, j \neq i. \end{aligned} \tag{17}$$

Based on the CLF (16), we obtain the feedback laws

$$\begin{aligned} u_i=&\left(Z_i(w)-U_i'(y_i)-\sum_{j\neq i}p_{i,j}V_{i,j}'(d_{i,j})\frac{(y_i-y_j)}{d_{i,j}}-b\sin(\theta_i)F_i\right) \\ &\times\left(v^*+\frac{A}{v_i\left(\cos(\theta_i)-\cos(\varphi)\right)^2}+v_i\cos(\theta_i)(b-1)\right)^{-1} \end{aligned} \tag{18}$$

$$F_i = -\frac{1}{\cos(\theta_i)}\left(k_i(w)\left(v_i\cos(\theta_i)-v^*\right)+\Lambda_i(w)\right) \tag{19}$$

where

$$\begin{aligned} Z_i(w) :=& -\mu_1 v_i\sin(\theta_i) \\ &+\sum_{j\neq i}\kappa_{i,j}(d_{i,j})\left(g_2(v_j\sin(\theta_j))-g_2(v_i\sin(\theta_i))\right) \end{aligned} \tag{20}$$

$$\begin{aligned} \Lambda_i(w) =& \sum_{j\neq i}V_{i,j}'(d_{i,j})\frac{(x_i-x_j)}{d_{i,j}} \\ &-\sum_{j\neq i}\kappa_{i,j}(d_{i,j})\left(g_1(v_j\cos(\theta_j))-g_1(v_i\cos(\theta_i))\right) \end{aligned} \tag{21}$$

$$k_i(w) = \mu_2+\frac{\Lambda_i(w)}{v^*}+\frac{v_{\max}\cos(\theta_i)}{v^*(v_{\max}\cos(\theta_i)-v^*)}r\left(-\Lambda_i(w)\right) \tag{22}$$

and $\mu_1, \mu_2 > 0$ are constants (controller gains), and $r \in C^1(\mathbb{R})$, $g_j \in C^1(\mathbb{R})$, $j = 1, 2$, are functions that satisfy

$$\max(0, x) \leq r(x) \quad \text{for all } x \in \mathbb{R} \tag{23}$$

$$g_j'(x) > 0 \quad \text{for } x \in \mathbb{R},\ j = 1, 2. \tag{24}$$

The term $k_i(w)$ in the acceleration $F_i(t)$ given by (22), is a state-dependent controller gain, which guarantees that the speed of each vehicle will remain positive and less than the speed limit. The first term in (19) drives the longitudinal speed of a vehicle towards the speed set-point v^*. If $V_{i,j}$ in (8), (9) is monotone, then, if vehicle j is behind vehicle i, i.e., $(x_i - x_j) > 0$, we have that $-V_{i,j}'(d_{i,j})\frac{(x_i-x_j)}{d_{i,j}} > 0$, and this term represents the effect of nudging [17], since vehicles that are close and behind vehicle i will exert on it a "pushing" force that increases its acceleration. It should be noted that properties (9) and (13) guarantee that the feedback laws (18) and (19), are *decentralized* and depend only on the relative speeds and distances from adjacent vehicles, more precisely from vehicles that are located at a distance less than $\lambda > 0$.

When the NCC is inviscid, then it does not require measurement of the speeds of the adjacent vehicles.

C. Pseudo-Relativistic Cruise Controller (PRCC)

The CLF in this case is given by

$$\begin{aligned} H_R(w) :=& \frac{1}{2}\sum_{i=1}^{n}\frac{(v_i\cos(\theta_i)-v^*)^2+bv_i^2\sin^2(\theta_i)}{(v_{\max}-v_i)\,v_i} \\ &+\sum_{i=1}^{n}U_i(y_i)+\frac{1}{2}\sum_{i=1}^{n}\sum_{i\neq j}V_{i,j}(d_{i,j}) \\ &+A\sum_{i=1}^{n}\left(\frac{1}{cos(\theta_i)-\cos(\varphi)}-\frac{1}{1-\cos\varphi}\right) \end{aligned} \tag{25}$$

where $A > 0$, $b > 1 - \frac{v^*}{v_{\max}} > 0$ (recall that $v^* \in (0, v_{\max})$) are parameters of the controller and the Lyapunov function. Notice that the kinetic energy term in H_R (i.e., the term $\frac{1}{2}\sum_{i=1}^{n} \frac{(v_i \cos(\theta_i) - v^*)^2 + b v_i^2 \sin^2(\theta_i)}{(v_{\max} - v_i) v_i}$) is not related to the kinetic energy of classical mechanics, but is similar to the kinetic energy of a system of n particles in relativistic mechanics, with speed limits 0 and $v_{\max}$ in place of $-c$ and c, where c is the speed of light, which are the speed limits in relativistic mechanics. In relativistic mechanics, the kinetic energy increases to infinity when the speed of an object approaches (in absolute value) the speed of light, which indicates that no object with mass can reach the speed of light. Analogously, in (25), the kinetic energy term grows to infinity as the speed of a vehicle approaches zero or the maximum speed $v_{\max}$, thus restricting the speed of vehicles in $(0, v_{\max})$. As in the case of (16), the last terms represent the potential energy of the system and a penalty term that blows up when $\theta_i \to \pm\varphi$.

The following result shows that the function H_R is a size function for the state space Ω defined by (7).

Lemma 2: *Let constants $A > 0$, $v_{\max} > 0$, $v^* \in (0, v_{\max})$, $L_{i,j} > 0$, $i, j = 1, ..., n$, $i \neq j$, $\lambda > 0$ that satisfies* (15)*, $\varphi \in \left(0, \frac{\pi}{2}\right)$ that satisfies* (4)*, and define the function $H_R : \Omega \to \mathbb{R}_+$ by means of* (25)*, where Ω is given by* (7)*. Then, there exist non-decreasing functions $\eta_i : \mathbb{R}_+ \to [0, a)$, $i = 1, ..., n$, $\ell_2 : \mathbb{R}_+ \to [v^*, v_{\max})$, $\omega : \mathbb{R}_+ \to [0, \varphi)$, a non-increasing function $\ell_1 : \mathbb{R}_+ \to (0, v^*]$, and, for each pair $i, j = 1, ..., n$, $i \neq j$, there exist non-increasing functions $\rho_{i,j} : \mathbb{R}_+ \to (L_{i,j}, \lambda]$ with $\rho_{i,j}(s) \equiv \rho_{j,i}(s)$, such that the following implications hold for all $i, j = 1, ..., n$, $j \neq i$:*

$$w \in \Omega \Rightarrow \ell_1 (H_R(w)) \le v_i \le \ell_2 (H_R(w)), |\theta_i| \le \omega (H_R(w)), \quad |y_i| \le \eta_i (H_R(w)), d_{i,j} \ge \rho_{i,j} (H_R(w)), \tag{26}$$

Let $f_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2$, be C^1 functions that satisfy:

$$f_j(0) = 0 \text{ and } x\, f_j(x) > 0, \text{ for } x \neq 0, \ j = 1, 2 \tag{27}$$

and $g_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2$, be C^1 functions that satisfy (24). The controllers that correspond to the CLF (25) are

$$u_i = \frac{v_i}{\beta(v_i, \theta_i)} \Bigg(G_i(w) - U_i'(y_i) - a(v_i, \theta_i) F_i - \sum_{j \neq i} p_{i,j} V_{i,j}'(d_{i,j}) \frac{(y_i - y_j)}{d_{i,j}} \Bigg) \tag{28}$$

$$F_i = \frac{1}{q(v_i, \theta_i)} \left(R_i(w) - \sum_{j \neq i} V_{i,j}'(d_{i,j}) \frac{(x_i - x_j)}{d_{i,j}} \right) \tag{29}$$

where

$$G_i(w) = -f_2 (v_i \sin(\theta_i)) + \sum_{j \neq i} \kappa_{i,j} (d_{i,j}) \left(g_2 (v_j \sin(\theta_j)) - g_2 (v_i \sin(\theta_i)) \right) \tag{30}$$

$$R_i(w) = -f_1 (v_i \cos(\theta_i) - v^*) + \sum_{j \neq i} \kappa_{i,j} (d_{i,j}) \left(g_1 (v_j \cos(\theta_j)) - g_1 (v_i \cos(\theta_i)) \right) \tag{31}$$

and

$$q(v, \theta) := \frac{v_{\max} v \cos(\theta) + v^* v_{\max} - 2 v^* v}{2 (v_{\max} - v)^2 v^2} \tag{32}$$

$$\beta(v, \theta) := \frac{A}{(\cos(\theta) - \cos(\varphi))^2} + \frac{(b-1) v \cos(\theta) + v^*}{(v_{\max} - v)} \tag{33}$$

$$a(v, \theta) := \frac{b v_{\max} \sin(\theta)}{2 (v_{\max} - v)^2 v}. \tag{34}$$

Notice that the definition of b implies that $\beta(v, \theta) > 0$ for all $v \in (0, v_{\max})$, and $\theta \in (-\varphi, \varphi)$.

The pseudo-relativistic feedback laws (28) and (29) are derived by using the CLF (25), which is also a size function. Compared to the Newtonian controller (18), (19), the controller (28), (29) is simpler, since it does not use state-dependent controller gains to restrict the speed in $(0, v_{\max})$ (due to the properties of the size function H_R). The function $\frac{1}{q(v,\theta)}$ in (29), (32) drives the acceleration F_i to zero, when the speed of the vehicle tends to zero or to the maximum speed $v_{\max}$. Finally, notice that properties (9) and (13) guarantee that the feedback laws (28) and (29), are *decentralized* (per vehicle) and depend only on the relative speed of and distance from adjacent vehicles, namely vehicles that are located at a distance less than $\lambda > 0$. Again, it should be noted that, when the PRCC is inviscid, then it does not require measurement of the speeds of the adjacent vehicles.

D. Main Results

The following theorem shows that each of the closed-loop systems (2), (18), (19), and (2), (28), (29) satisfy properties (P1) and (P2).

Theorem 1 (Closed-loop system with NCC): *For every $w_0 \in \Omega$ there exists a unique solution $w(t) \in \Omega$ of the initial-value problem* (2), (18), (19) *with initial condition $w(0) = w_0$. The solution $w(t) \in \Omega$ is defined for all $t \ge 0$ and satisfies*

$$\lim_{t \to +\infty} (v_i(t)) = v^*, \lim_{t \to +\infty} (\theta_i(t)) = 0 \tag{35}$$

$$\lim_{t \to +\infty} (u_i(t)) = 0, \lim_{t \to +\infty} (F_i(t)) = 0. \tag{36}$$

Moreover, there exist non-decreasing functions $Q_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, 2)$ such that: $\max_{i=1,...,n} (|F_i(t)|) \le Q_1(H(w(0)))$, and $\max_{i=1,...,n} (|u_i(t)|) \le Q_2(H(w(0)))$, for all $t \ge 0$, for every solution $w(t) \in \Omega$ of (2), (18), (19).

Theorem 2 (Closed-loop system with PRCC): *For every $w_0 \in \Omega$ there exists a unique solution $w(t) \in \Omega$ of the initial-value problem* (2), (28), (29) *with initial condition $w(0) = w_0$. The solution $w(t) \in \Omega$ is defined for all $t \ge 0$ and satisfies* (35) *and* (36) *for all $i = 1, ..., n$. Moreover, there exist non-decreasing functions $Q_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 3, 4)$ such that $\max_{i=1,...,n} (|F_i(t)|) \le Q_3(H_R(w(0)))$, for all $t \ge 0$ and $\max_{i=1,...,n} (|u_i(t)|) \le Q_4(H_R(w(0)))$, for all $t \ge 0$ for every solution $w(t) \in \Omega$ of* (2), (28), (29).

Remarks: (i) The results of Theorem 1 and Theorem 2 hold globally, i.e., for any initial condition $w_0 \in \Omega$.

(ii) Simulations have showed that, by increasing viscosity we manage to increase the convergence rate of the vehicles speeds to the speed set point v^* for the closed-loop system with both NCC and PRCC. However, the convergence rate is not exponential (see the analysis in [13]).

IV. Macroscopic Models

In this section we present the macroscopic models that correspond to the microscopic model (2) with the NCC (18), (19) or the PRCC (28), (29).

Let $\rho_{\max}, v_{\max} > 0$ and $v^* \in (0, v_{\max})$, $\bar{\rho} \in (0, \rho_{\max})$ be constants and let $\mu : (0, \rho_{\max}) \to \mathbb{R}_+$, $P : (0, \rho_{\max}) \to \mathbb{R}_+$ be $C^2((0, \rho_{\max}))$ and non-negative functions that satisfy:

$$\limsup_{\rho \to \rho_{\max}^-} (P(\rho)) = +\infty, \tag{37}$$

$$\mu(\rho) = 0,\ P(\rho) = 0 \text{ for all } \rho \in (0, \bar{\rho}] \tag{38}$$

The macroscopic model that corresponds to the microscopic model (2) under the PRCC (28), (29) is the following system of PDEs for $t > 0$ and $x \in I(t)$, where $I(t) \subseteq \mathbb{R}$ is an appropriate interval

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{39}$$

$$q(v)\frac{\partial v}{\partial t} + q(v) v \frac{\partial v}{\partial x} + \frac{P'(\rho)}{\rho}\frac{\partial \rho}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} g'(v)\mu(\rho)\right) - f(v - v^*) \tag{40}$$

with constraints $0 < \rho(t,x) < \rho_{\max}$, $0 < v(t,x) < v_{\max}$ for all $t > 0$ and $x \in I(t)$, where $g \in C^1(\mathbb{R})$ is an increasing function with $g'(v) > 0$ for all $v \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ is a C^1 function with $x f(x) > 0$ for all $x \neq 0$ and

$$q(v) := \frac{v_{\max} v - 2v^* v + v^* v_{\max}}{2(v_{\max} - v)^2 v^2}. \tag{41}$$

The term $f(v - v^*)$ is a *relaxation term* that describes the tendency of vehicles to adjust their speed to the speed set-point v^* and is similar to friction, see [6]. The term $\frac{P'(\rho)}{\rho}\frac{\partial \rho}{\partial x}$ is a *pressure term* and expresses the tendency to accelerate or to decelerate based on the (local) density. Finally, the term $\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} g'(v)\mu(\rho)\right)$, is a *viscosity term,* by analogy with the theory of fluids, with $\mu(\rho)$ playing the role of dynamic viscosity. When $g(v) \equiv v$, the viscosity term is exactly the same as the viscosity term appearing in compressible fluid flow (see [14], [15], [19] and references therein). When $g(v)$ does not coincide with v, then the viscosity term is similar to the viscosity term appearing in porous fluid flow (see [1]).

The macroscopic model that corresponds to the NCC is given by the continuity equation (39) and the following momentum equation for $t > 0$ and $x \in I(t)$, where $I(t) \subseteq \mathbb{R}$ is an appropriate interval

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{P'(\rho)}{\rho}\frac{\partial \rho}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} g'(v)\mu(\rho)\right) - (\gamma + h(G))(v - v^*) \tag{42}$$

where

$$G = -\frac{P'(\rho)}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} g'(v)\mu(\rho)\right)$$

$$h(s) = \frac{v_{\max} r(s)}{v^*(v_{\max} - v^*)} - \frac{s}{v^*}$$

and $r \in C^1$ satisfies (23). Again, the macroscopic model (39), (42) is to be considered with constraints $0 < \rho(t,x) < \rho_{\max}$, $0 < v(t,x) < v_{\max}$ for all $t > 0$ and $x \in I(t)$, with $g \in C^1(\mathbb{R})$ being an increasing function with $g'(v) > 0$ for all $v \in \mathbb{R}$ and $\gamma > 0$ being a constant.

Remarks: (i) Both models (39), (40) and (39), (42) do not include non-local terms and have certain characteristics from the kinematic theory of fluids. Traffic flow is isotropic, as in fluid flow, since the vehicles are affected by both upstream and downstream vehicles, due to the nudging induced by the NCCs and PRCCs.

(ii) There are infinite equilibria for both models, namely the points where $v(x) \equiv v^*$ and $\rho(x) \leq \bar{\rho}$ for all $x \in \mathbb{R}$.

(iii) For the inviscid NCC-based model ($\mu(s) \equiv 0$), it was shown in [13], that, if the density is sufficiently small, then the solution of the macroscopic model approaches the equilibrium speed (in the sup norm); while the density converges exponentially to a traveling wave.

(iv) the model (42) is highly nonlinear due to the presence of a highly nonlinear relaxation term $(\gamma + h(G))(v - v^*)$ in the speed PDE.

The macroscopic models (39), (40) and (39), (42) can approximate the movement of n identical vehicles with total mass $m > 0$ on a straight road under the PRCC and under the NCC, when the following assumption holds:

Assumption 1: (i) the vehicles are constrained to move on a line (longitudinal motion),
(ii) there exist constants $\lambda > L > 0$ with $\lambda < 2L$ such that $V_{i,j}(s) = \Phi(ns)$ for all $i, j = 1, ..., n$ and $s > L/n$, where $\Phi : (L, +\infty) \to \mathbb{R}_+$ is a C^2 function that satisfies $\lim_{d \to L^+} (\Phi(d)) = +\infty$ and $\Phi(d) = 0$ for all $d \geq \lambda$,
(iii) $\kappa_{i,j}(s) = n^2 K(ns)$, for all $i, j = 1, ..., n$ and $s > L/n$, where $K : (L, +\infty) \to \mathbb{R}_+$ is a C^1 function that satisfies $K(d) = 0$ for all $d \geq \lambda$, and
(iv) the number of vehicles n is very large (tends to infinity).

We next present the relations between the various parameters and functions involved in the NCC and PRCC on one hand; and the corresponding macroscopic quantities involved in the macroscopic models on the other hand. Table 1 shows how all parameters and functions of the macroscopic models can be directly obtained from the corresponding microscopic models.

Notice that the pressure P is based on the derivative of the potential Φ, which, in the microscopic case, exerts the same force to the following and preceding vehicle (nudging). The latter is in analogy with fluids where pressure at any point in a fluid is (locally) the same in all directions. The dynamic viscosity $\mu(\rho)$ makes the "traffic fluid" act as a Newtonian fluid. However, in contrast to actual fluids, the "traffic fluid" induced by the PRCCs or the NCCs satisfies (38). The function $g(v)$ is met in various nonlinear PDEs. For fluid flows in non-porous media, we have $g(v) \equiv v$. For compressible fluid flows in porous media, the function $g(v)$ takes the form $g(v) = cv^m$, where $c, m > 0$ are constants

	Macroscopic	Microscopic
Maximum Density	$\rho_{\max}$	$= \frac{m}{L}$
Maximum Speed	v_{max}	$= v_{\max}$
Desired Speed	v^*	$= v^*$
Interaction Density	$\bar{\rho}$	$= \frac{m}{\lambda}$
Dynamic Viscosity	$\mu(\rho)$	$= \frac{m^2}{\rho} K\left(\frac{m}{\rho}\right)$
	$g(v)$	$= g_1(v)$
Pressure	$P(\rho)$	$= z - m\Phi'\left(\frac{m}{\rho}\right)$, arbitrary $z \in \mathbb{R}$
Relaxation Term (PRCC)	$f(v - v^*)$	$= f_1(v - v^*)$
Relaxation Term (NCC)	$r(x)$	$= r(x)$
Constant Parameter (NCC)	γ	$= \mu_2$

TABLE I: Relations between macroscopic and microscopic quantities

(see [1], but notice also the essential difference that, for flow in porous media, the function g is a function of the density ρ instead of the speed v). The relaxation terms appear as friction terms (see [6], but, instead of penalizing the speed v, the relaxation terms penalize the deviation of the speed from the desired speed v^*).

The most important implication from Table 1 is the fact that, by changing the functions and the parameters of the NCCs and the PRCCs, *we can directly determine the physical properties of the "traffic fluid"*. In this sense, we may talk about an engineered or designed artificial fluid that approximates the actual emerging traffic flow. To understand how far the implications of the relations between the cruise controller parameters and the characteristics of the traffic fluid go, it is important to notice that for isentropic (or barotropic) flow of gases, the dynamic viscosity and the pressure are always increasing functions of the fluid density (see the discussion in [14]). However, for a traffic fluid that emerges from the use of NCCs or PRCCs, if the cruise controller uses a non-monotone potential function Φ or a non-monotone viscosity function K, then it is possible to obtain a traffic fluid with dynamic viscosity and pressure, which are non-monotone functions of the fluid density (and can have local minima). Thus, the traffic fluid can be arranged to have very different physical properties from those of real compressible fluids (gases).

V. Conclusions

In the present work, we have presented CLF-based cruise controllers for the safe operation of autonomous vehicles on lane-free roads base. The CLF were based on measures of the total energy of the system. By expressing the kinetic energy as in Newtonian or relativistic mechanics, two families of decentralized (per vehicle) controllers were obtained that guarantee the safe operation of vehicles on lane-free roads. Moreover, we have presented the corresponding macroscopic models consisting of a conservation equation and a momentum equation with pressure and viscous terms. By selecting appropriately the parameters of the cruise controllers, we can directly influence the physical characteristics of the "traffic fluid", thus creating an artificial fluid that approximates the traffic flow.

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